Homework No. 03 (Spring 2015)

PHYS 420: Electricity and Magnetism II

Due date: Monday, 2015 Feb 23, 4.30pm

1. (**10 points.**) Using Mathematica (or another graphing tool) plot the Legendre polynomials $P_l(x)$ for $l = 0, 1, 2, 3, 4$ on the same plot. Based on the pattern you see what can you conclude about the number of roots for $P_l(x)$.

2. (**10 points.**) The induced charge on the surface of a spherical conducting shell of radius a due to a point charge q placed a distance b away is given by

$$\rho(\mathbf{r}) = \sigma(\theta, \phi)\,\delta(r - a), \tag{1}$$

where

$$\sigma(\theta, \phi) = -\frac{q}{4\pi a} \frac{(r_>^2 - r_<^2)}{(a^2 + b^2 - 2ab\cos\theta)^{\frac{3}{2}}}, \tag{2}$$

where $r_< = \text{Min}(a, b)$ and $r_> = \text{Max}(a, b)$. Calculate the dipole moment of this charge configuration (excluding the original charge q) using

$$\mathbf{d} = \int d^3r\,\mathbf{r}\,\rho(\mathbf{r}), \tag{3}$$

for the two cases $a < b$ and $a > b$, representing the charge being inside or outside the sphere.

3. (**40 points.**) Recollect Legendre polynomials

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2 - 1)^l}{2^l l!}. \tag{4}$$

In particular

$$P_0(x) = 1, \tag{5a}$$
$$P_1(x) = x, \tag{5b}$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}. \tag{5c}$$

Consider a charged spherical shell of radius a consisting of a charge distribution in the polar angle alone,

$$\rho(\mathbf{r}') = \sigma(\theta')\,\delta(r' - a). \tag{6}$$

The electric potential *on the z-axis*, $\theta = 0$ and $\phi = 0$, is then given by

$$\phi(r,0,0) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$
$$= \frac{2\pi a^2}{4\pi\varepsilon_0}\int_0^\pi \sin\theta' d\theta' \frac{\sigma(\theta')}{\sqrt{r^2+a^2-2ar\cos\theta'}}, \tag{7}$$

after evaluating the r' and ϕ' integral.

(a) Consider a uniform charge distribution on the shell,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_0(\cos\theta). \tag{8}$$

Evaluate the integral in Eq. (7) to show that

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_>}, \tag{9}$$

where $r_< = \mathrm{Min}(a,r)$ and $r_> = \mathrm{Max}(a,r)$.
Note: This was done in class. Nevertheless, present the relevant steps.

(b) Next, consider a (pure dipole, 2×1-pole,) charge distribution of the form,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_1(\cos\theta). \tag{10}$$

Evaluate the integral in Eq. (7) to show that

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{3}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right). \tag{11}$$

Note: This was done in class. Nevertheless, present the relevant steps.

(c) Next, consider a (pure quadrapole, 2×2-pole,) charge distribution of the form,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_2(\cos\theta). \tag{12}$$

Evaluate the integral in Eq. (7) to show that

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{5}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^2. \tag{13}$$

(d) For a (pure $2l$-pole) charge distribution

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_l(\cos\theta) \tag{14}$$

the integral in Eq. (7) leads to

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{(2l+1)}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l. \tag{15}$$

Note: No work needs to be submitted for this part. We will prove this in class.