Homework No. 09 (Spring 2015)

PHYS 420: Electricity and Magnetism II

Due date: Friday, 2015 Apr 24, 4.30pm

1. (**20 points.**) Lorentz transformation (in one dimension) is given by

$$\Delta z' = \gamma(\Delta z - v\Delta t), \tag{1a}$$
$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c}\frac{\Delta z}{c}\right), \tag{1b}$$

where $\gamma = \sqrt{1 - v^2/c^2}$. Show that for

$$v \ll c \qquad \text{and} \qquad \frac{\Delta z}{\Delta t} \ll c \tag{2}$$

one obtains the Galilean transformation

$$\Delta z' = \Delta z - v\Delta t, \tag{3a}$$
$$\Delta t' = \Delta t. \tag{3b}$$

2. (**20 points.**) Lorentz transformation describing a boost in the x-direction, y-direction, and z-direction, are

$$L_1 = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} \gamma_2 & 0 & -\beta_2\gamma_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_2\gamma_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} \gamma_3 & 0 & 0 & -\beta_3\gamma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_3\gamma_3 & 0 & 0 & \gamma_3 \end{pmatrix}, \tag{4}$$

respectively. Transformation describing a rotation about the x-axis, y-axis, and z-axis, are

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\omega_1 & \sin\omega_1 \\ 0 & 0 & -\sin\omega_1 & \cos\omega_1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_2 & 0 & -\sin\omega_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\omega_2 & 0 & \cos\omega_2 \end{pmatrix}, \qquad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_3 & \sin\omega_3 & 0 \\ 0 & -\sin\omega_3 & \cos\omega_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

respectively. For infinitesimal transformations use the approximations

$$\gamma_i \sim 1, \qquad \cos\omega_i \sim 1, \qquad \sin\omega_i \sim \omega_i, \tag{6}$$

to identify the generator for boosts $\mathbf{N}$, and the generator for rotations the angular momentum $\mathbf{J}$,

$$\mathbf{L} = \mathbf{1} + \boldsymbol{\beta}\cdot\mathbf{N} \qquad \text{and} \qquad \mathbf{R} = \mathbf{1} + \boldsymbol{\omega}\cdot\mathbf{J}, \tag{7}$$

respectively. Then derive

$$\Big[N_1, N_2\Big] = N_1 N_2 - N_2 N_1 = J_3. \tag{8}$$

This states that boosts in perpendicular direction leads to rotation. (To gain insight of the statement, calculate $[J_1, J_2]$ and interpret the result.)

3. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration α is described by the equation of a hyperbola

$$x^2 - c^2 t^2 = x_0^2, \qquad x_0 = \frac{c^2}{\alpha}. \tag{9}$$

This is the motion of a particle 'dropped' from $x = x_0$ at $t = 0$ in region of constant (proper) acceleration.

(a) Will a photon dispatched to 'chase' this particle at $t = 0$ from $x = 0$ ever catch up with it? If yes, when and where does it catch up?

(b) Will a photon dispatched to 'chase' this particle at $t = 0$ from $0 < x < x_0$ ever catch up with it? If yes, when and where does it catch up?

(c) Will a photon dispatched to 'chase' this particle, at $t = 0$ from $x < 0$ ever catch up with it? If yes, when and where does it catch up?

What are the implications for the observable part of our universe from this analysis?