Homework No. 10 (Spring 2015)

PHYS 420: Electricity and Magnetism II

Due date: Wednesday, 2015 May 6, 4.30pm

1. (**20 points.**) Using Maxwell's equations, without introducing potentials, show that the electric and magnetic fields satisfy the inhomogeneous wave equations

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(\mathbf{r},t) = -\frac{1}{\varepsilon_0}\boldsymbol{\nabla}\rho(\mathbf{r},t) - \frac{1}{\varepsilon_0}\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t), \tag{1a}$$

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B}(\mathbf{r},t) = \mu_0\boldsymbol{\nabla}\times\mathbf{J}(\mathbf{r},t). \tag{1b}$$

2. (**20 points.**) Consider the retarded Green's function

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t-t'-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right). \tag{2}$$

(a) For $\mathbf{r}'=0$ and $t'=0$ show that

$$G(r,t) = \frac{1}{r}\,\delta\left(t-\frac{r}{c}\right). \tag{3}$$

(b) Then, evalauate

$$\int_{-\infty}^{\infty} dt\, G(r,t). \tag{4}$$

(c) From the answer above, what can you comment on the physical interpretation of $\int_{-\infty}^{\infty} dt\, G(r,t)$.

3. (**20 points.**) Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{5}$$

where the sum on r runs over the roots a_r of the equation $F(x)=0$, evaluate

$$\delta(ax^2+bx+c). \tag{6}$$

4. (**20 points.**) Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{7}$$

where the sum on r runs over the roots a_r of the equation $F(x)=0$, evaluate

$$\delta(\sin x). \tag{8}$$

5. (**20 points.**) A charged particle with charge q moves on the z-axis with constant speed v, $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$. The scalar and vector potential generated by this charged particle is

$$\phi(\mathbf{r},t) = \gamma \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{(x^2+y^2)+\gamma^2(z-vt)^2}}, \tag{9a}$$

$$c\mathbf{A}(\mathbf{r},t) = \beta\gamma \frac{q}{4\pi\varepsilon_0} \frac{\hat{\mathbf{z}}}{\sqrt{(x^2+y^2)+\gamma^2(z-vt)^2}}. \tag{9b}$$

(a) Using

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial}{\partial t}\mathbf{A}, \tag{10a}$$

$$\mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{10b}$$

evaluate the electric and magnetic field generated by the charged particle to be

$$\mathbf{E}(\mathbf{r},t) = \gamma \frac{q}{4\pi\varepsilon_0} \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z-vt)\hat{\mathbf{k}}}{[(x^2+y^2)+\gamma^2(z-vt)^2]^{\frac{3}{2}}}, \tag{11a}$$

$$c\mathbf{B}(\mathbf{r},t) = \beta\gamma \frac{q}{4\pi\varepsilon_0} \frac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{[(x^2+y^2)+\gamma^2(z-vt)^2]^{\frac{3}{2}}}. \tag{11b}$$

(b) Evaluate the electromagnetic momentum density for this configuration by evaluating

$$\mathbf{G}(\mathbf{r},t) = \varepsilon_0 \mathbf{E}(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t). \tag{12}$$