Date: ~~2015 Jan 23~~
2015 Jan 26

(1) Electrostatics is governed by

$$\vec{\nabla} \cdot \epsilon_0 \vec{E} = \underbrace{\rho(\vec{r})}_{\text{charges}} - \underbrace{\vec{\nabla} \cdot \vec{P}}_{\text{polarization of the material}}$$

and

$$\vec{\nabla} \times \vec{E} = 0,$$

$\vec{E} = -\vec{\nabla}\phi$

$\epsilon \vec{E} = \epsilon_0 \vec{E} + \vec{P}$

which together state

$$-\vec{\nabla} \cdot \underbrace{\epsilon(\vec{r})}_{\text{material}} \vec{\nabla} \phi(\vec{r}) = \underbrace{\rho(\vec{r})}_{\text{charges}}.$$

(2) The scalar potential $\phi(\vec{r})$ is expressed in terms of ~~the~~ Green's function by

$$\phi(\vec{r}) = \int d^3r' \, G(\vec{r}, \vec{r}') \, \rho(\vec{r}')$$

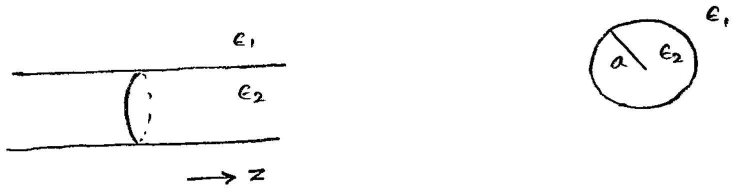
where ~~the~~ Green's function satisfies

$$-\vec{\nabla} \cdot \epsilon(\vec{r}) \vec{\nabla} G(\vec{r}, \vec{r}') = \delta^{(3)}(\vec{r} - \vec{r}').$$

In this sense ~~the~~ Green's function is interpreted as the potential due to a unit point charge at position $\vec{r}'$ in the presence of the material described by $\epsilon(\vec{r})$.

(3) We would like to consider a circular cylinder described by its dielectric property,

$$\epsilon(\vec{r}) = \begin{cases} \epsilon_2 , & 0 \leqslant \rho < a , \\ \epsilon_1 , & 0 < a < \rho . \end{cases}$$



(4) But, let us first solve for the free Green's function (no dielectric present)

$$- \epsilon_0 \nabla^2 G(\vec{r}, \vec{r}') = \delta^{(3)}(\vec{r} - \vec{r}')$$

in terms of cylindrical coordinates (ρ, ϕ, z).

Note that we have earlier derived various representations for the Green function:

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}'|}$$

$$= \frac{1}{\epsilon_0} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{r} - \vec{r}')} \frac{1}{k^2}$$

$$= \frac{1}{\epsilon_0} \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r} - \vec{r}')_\perp} \frac{1}{2k_\perp} e^{-k_\perp |z - z'|}$$

(5) Requiring translational symmetry in z and rotational symmetry in ϕ we can write

$$G(\vec{r},\vec{r}') = G(\rho,\rho',\phi-\phi',z-z')$$
$$= \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi}\, e^{ik_z(z-z')} \cdot \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')}\, g_m(\rho,\rho';k_z)$$

(6) Substituting the above form for Green function in the differential equation in (4), after using

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

and

$$\delta^{(3)}(\vec{r}-\vec{r}') = \frac{\delta(\rho-\rho')}{\rho}\,\delta(z-z')\,\delta(\phi-\phi')$$
$$= \frac{\delta(\rho-\rho')}{\rho} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi}\, e^{ik_z(z-z')}\, \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')},$$

leads to the differential equation for the reduced Green's function $g_m(\rho,\rho';k)$

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{m^2}{\rho^2} + k^2\right] \epsilon_0\, g_m(\rho,\rho';k) = \frac{\delta(\rho-\rho')}{\rho}.$$

(7) Modified Bessel functions, $K_m(t)$ and $I_m(t)$, are defined using the integral representations,

$$K_m(t) = \int_0^\infty d\theta \; \cosh m\theta \; e^{-t\cosh\theta} ,$$

$$I_m(t) = \int_0^\pi \frac{d\phi}{\pi} \; \cos m\phi \; e^{-t\cos\phi} .$$

(8) The differential equation for the reduced Green's function, in (6), in source free regions $\rho \neq \rho'$, are satisfied by the modified Bessel functions. That is,

$$\left[-\frac{1}{t}\frac{d}{dt} t \frac{d}{dt} + \frac{m^2}{t^2} + 1 \right] \left\{ \begin{matrix} K_m(t) \\ I_m(t) \end{matrix} \right\} = 0 .$$

(9) Thus, for the regions $\rho \neq \rho'$, we can write

$$g_m(\rho, \rho'; k) = \begin{cases} A \; K_m(k\rho) + B \; I_m(k\rho) , & \rho < \rho', \\ C \; K_m(k\rho) + D \; I_m(k\rho) , & \rho' < \rho, \end{cases}$$

such that they satisfy the boundary conditions :

(i) $g_m(0, \rho'; k)$ is finite.

(ii) $g_m(\infty, \rho'; k)$ is finite.

(iii) $g_m \Big|_{\rho = \rho' - \delta}^{\rho = \rho' + \delta} = 0$ (g_m is continuous at $\rho = \rho'$.)

(iv) $\rho \frac{\partial}{\partial \rho} g_m \Big|_{\rho = \rho' - \delta}^{\rho = \rho' + \delta} = -\frac{1}{\varepsilon_0}$ (derivative is not continuous at $\rho = \rho'$.)

(10) The boundary condition (i) and (ii) imply,

$$A = 0,$$
$$D = 0.$$

And, the continuity conditions, after using $A=0$ and $D=0$, imply

$$C\, K_m(k\rho') - B\, I_m(k\rho') = 0,$$

$$C\, K_m'(k\rho') - B\, I_m'(k\rho') = -\frac{1}{\epsilon_0}\frac{1}{k\rho'}.$$

Thus, the coefficients are determined as

$$C = -\frac{1}{\epsilon_0}\frac{1}{k\rho'}\frac{I_m(k\rho')}{\left[I_m(k\rho')\, K_m'(k\rho') - I_m'(k\rho')\, K_m(k\rho')\right]}$$

$$B = -\frac{1}{\epsilon_0}\frac{1}{k\rho'}\frac{K_m(k\rho')}{\left[I_m(k\rho')\, K_m'(k\rho') - I_m'(k\rho')\, K_m(k\rho')\right]}.$$

(11) In homework you will derive the Wronskian

$$I_m(t)\, K_m'(t) - I_m'(t)\, K_m(t) = -\frac{1}{t}.$$

Hint: Multiply the differential equations in (8) by I_m and K_m, respectively, and subtract.

12 Using the Wronskian of 11 the coefficients simplify,

$$A = 0,$$

$$B = \frac{1}{\epsilon_0} K_m(k\rho'),$$

$$C = \frac{1}{\epsilon_0} I_m(k\rho'),$$

$$D = 0,$$

which in 9 gives

$$g_m(\rho, \rho'; k) = \begin{cases} \frac{1}{\epsilon_0} I_m(k\rho) K_m(k\rho'), & \rho < \rho', \\ \frac{1}{\epsilon_0} I_m(k\rho') K_m(k\rho), & \rho' < \rho, \end{cases}$$

$$= \frac{1}{\epsilon_0} I_m(k\rho_<) K_m(k\rho_>) \qquad \rho_< = \mathrm{Min}(\rho, \rho'), \quad \rho_> = \mathrm{Max}(\rho, \rho')$$

13 Thus, the free Green's function in cylindrical coordinates is given by

$$G(\vec{r}, \vec{r}') = \frac{1}{\epsilon_0} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik(z-z')} \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')} I_m(k\rho_<) K_m(k\rho_>).$$