Date: 2015 Jan 30

Interaction energy between a charge and a perfectly conducting cylinder

① For two charges we have $q_1 \cdot \quad \cdot q_2$

$$E_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$$

② For three charges we have

$$E_{int} = \frac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^{3} \sum_{j=1}^{3} \frac{1}{4\pi\epsilon_0} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

$i \neq j$: removes the self interaction terms

$\frac{1}{2}$: removes the double counting

③ For a continuous charge distribution the above generalizes to

$$E^0_{int} = \frac{1}{2} \int d^3r \int d^3r' \, \rho(\vec{r}) \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}'|} \rho(\vec{r}') - E_{self}$$

$$= \frac{1}{2} \int d^3r \int d^3r' \, \rho(\vec{r}) \, G_0(\vec{r}, \vec{r}') \, \rho(\vec{r}') - E_{self} ,$$

where $G_0(\vec{r}, \vec{r}')$ is the free Green's function, that is in the absence of materials, and E_{self} is the self energy.

(4) In the presence of a dielectric medium the Green's function is modified, and the interaction energy (associated with the interaction of charges) is accordingly modified to

$$E^{\epsilon}_{int} = \frac{1}{2}\int d^3r \int d^3r' \, \rho(\vec{r}) \, G_\epsilon(\vec{r},\vec{r}') \, \rho(\vec{r}') - E_{self},$$

which should be contrasted with

$$E^{0}_{int} = \frac{1}{2}\int d^3r \int d^3r' \, \rho(\vec{r}) \, G_0(\vec{r},\vec{r}') \, \rho(\vec{r}') - E_{self}.$$

(5) The change in energy of the system associated to the introduction of the material can thus be defined to be

$$\Delta E_{int} = E^{\epsilon}_{int} - E^{0}_{int}$$

$$= \frac{1}{2}\int d^3r \int d^3r' \, \rho(\vec{r}) \left[G_\epsilon(\vec{r},\vec{r}') - G_0(\vec{r},\vec{r}') \right] \rho(\vec{r}'),$$

which in this naive way gets rid of the self energy E_{self}.

(6) For a perfectly conducting cylinder we have

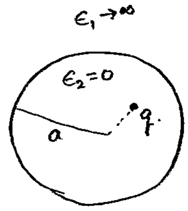
$$G_0(\vec{r},\vec{r}') = \frac{1}{\epsilon_0}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\, e^{ik(z-z')}\sum_{m=-\infty}^{+\infty}\frac{1}{2\pi}\, e^{im(\phi-\phi')}\, I_m(k\rho_<)\, K_m(k\rho_>)$$

$$G_\epsilon(\vec{r},\vec{r}') = \frac{1}{\epsilon_0}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\, e^{ik(z-z')}\sum_{m=-\infty}^{+\infty}\frac{1}{2\pi}\, e^{im(\phi-\phi')}$$

$$\times\left[\, I_m(k\rho_<)\, K_m(k\rho_>) - \frac{K_m(ka)}{I_m(ka)}\, I_m(k\rho)\, I_m(k\rho')\right]$$

(7) Thus, we have the interaction energy between a point charge and a perfectly conducting cylinder to be

$$\begin{aligned}\Delta E_{int} &= E_{int}^{\infty} - E_{int}^{0}\\ &= \frac{1}{2}\int d^3r\int d^3r'\; \rho(\vec{r})\left[\, G_\infty(\vec{r},\vec{r}') - G_0(\vec{r},\vec{r}')\right]\rho(\vec{r}')\\ &= \frac{1}{2}\, q^2\left[\, G_\infty(\vec{r}_0,\vec{r}_0) - G_0(\vec{r}_0,\vec{r}_0)\right],\end{aligned}$$



where we used

$$\rho(\vec{r}) = q\,\delta^{(3)}(\vec{r}-\vec{r}_0) \qquad \vec{r}_0 = (\rho_0, \phi_0, z_0)$$

$\vec{r}_0$ being the position of charge q.

8 Calculating the difference in the Green function using 6 and substituting in 7 we have the interaction energy

$$\Delta E_{int} = -\frac{q^2}{2\epsilon_0}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\, e^{\overbrace{ik(z_0-z_0)}^{=0}}\sum_{m=-\infty}^{+\infty}\frac{1}{2\pi}\, e^{\overbrace{im(\phi-\phi_0)}^{=0}}\,\frac{K_m(ka)}{I_m(ka)}\, I_m(k\rho_0)\, I_m(k\rho_0)$$

$$= -\frac{q^2}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\sum_{m=-\infty}^{+\infty}\frac{K_m(ka)}{I_m(ka)}\left[I_m(k\rho_0)\right]^2$$

where ρ_0 is the distance of the point charge from the center of the cylinder.

9 For the case the charge is on the central axis of the cylinder, we have $\rho_0 = 0$, which leads to

$$I_m(t) = \begin{cases} 1, & m=0 \\ 0, & m\neq 0 \end{cases}$$

$$\Delta E_{int} = -\frac{q^2}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\,\frac{K_m(ka)}{I_m(ka)}$$

$$= -\frac{q^2}{4\pi\epsilon_0}\,\frac{1}{2\pi a}\int_{-\infty}^{+\infty}dt\,\frac{K_m(t)}{I_m(t)}$$

$$= -\frac{q^2}{4\pi\epsilon_0}\,\frac{1.36768}{\pi a}.$$

$$\int_0^{\infty}dt\,\frac{K_m(t)}{I_m(t)} \approx 1.36768$$