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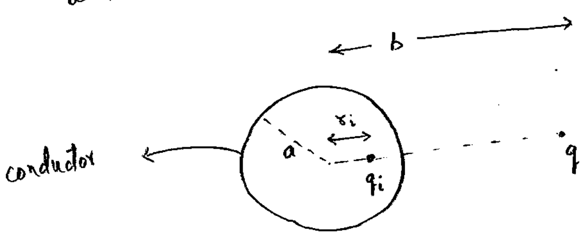
Induced charge density on a perfectly conducting spherical shell due to a point charge.

① We learned that the potential due to a point charge near a perfectly conducting spherical shell is as well described by the charge and its image charge.



$$q_i = -q\,\frac{a}{b}$$

$$r_i = a\,\frac{a}{b}$$

② Thus, the potential is given by

$$\phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{b}|} + \frac{q_i}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}_i|}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2+b^2-2rb\cos\theta}} - \frac{q}{4\pi\epsilon_0}\frac{\frac{a}{b}}{\sqrt{r^2+a^2\frac{a^2}{b^2}-2ra\frac{a}{b}\cos\theta}}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2+b^2-2rb\cos\theta}} - \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{\frac{b^2}{a^2}r^2+a^2-2rb\cos\theta}}$$

③ Clearly,

$$\phi(\vec{a}) = 0,$$

$\vec{a}$ being a vector on the surface of the spherical shell.

④ The charge density on the surface of a conductor is given by

$$\sigma(\theta,\phi) = \epsilon_0 \; \hat{n}\cdot\vec{E}\Big|_{surface}$$

$\hat{n} \rightarrow$ normal vector to the surface.

$$= \epsilon_0 \; \hat{r}\cdot\vec{E}(\vec{a})$$

$$= -\epsilon_0 \frac{\partial}{\partial r}\phi(\vec{r})\Big|_{r=a}.$$

⑤ Evaluating the derivative of the potential in ② we obtain

$$\sigma(\theta,\phi) = +\frac{q}{4\pi}\frac{(r - b\cos\theta)}{(r^2+b^2-2rb\cos\theta)^{\frac{3}{2}}} - \frac{q}{4\pi}\frac{\left(\frac{b^2}{a^2}r - b\cos\theta\right)}{\left(\frac{b^2}{a^2}r^2+a^2-2rb\cos\theta\right)^{\frac{3}{2}}}\Bigg|_{r=a}.$$

$$= \frac{q}{4\pi}\frac{(a - b\cos\theta)}{(a^2+b^2-2ab\cos\theta)^{\frac{3}{2}}} - \frac{q}{4\pi}\frac{\left(\frac{b^2}{a^2}a - b\cos\theta\right)}{(a^2+b^2-2ab\cos\theta)^{\frac{3}{2}}}$$

$$= \frac{q}{4\pi}\frac{a\left(1-\frac{b^2}{a^2}\right)}{(a^2+b^2-2ab\cos\theta)^{\frac{3}{2}}}$$

$$= -\frac{q}{4\pi a}\frac{(b^2-a^2)}{(a^2+b^2-2ab\cos\theta)^{\frac{3}{2}}}$$

6 The total amount of induced charge on the sphere is obtained as

$$\sigma_{tot} = a^2 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \;\; \sigma(\theta,\phi)$$

$$= a^2 \int_0^{\pi} \sin\theta \, d\theta \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \; (-1) \frac{q}{4\pi a} \frac{(b^2-a^2)}{(a^2+b^2-2ab\cos\theta)^{\frac{3}{2}}}$$

$$= -q \; \frac{a}{2} (b^2-a^2) \int_0^{\pi} \sin\theta \, d\theta \; \frac{1}{(a^2+b^2-2ab\cos\theta)^{\frac{3}{2}}}$$

$\cos\theta = t$
$-\sin\theta \, d\theta = dt$

$$= -q \; \frac{a}{2} (b^2-a^2) \int_{-1}^{1} dt \; \frac{1}{(a^2+b^2-2abt)^{\frac{3}{2}}}$$

$a^2+b^2-2abt = y$
$-2ab\,dt = dy$

$$= -q \; \frac{a}{2} (b^2-a^2) \int_{(b-a)^2}^{(b+a)^2} \frac{dy}{2ab} \; \frac{1}{y^{\frac{3}{2}}}$$

$$\int \frac{dy}{y^{\frac{3}{2}}} = \frac{y^{-\frac{3}{2}+1}}{-\frac{1}{2}} = -\frac{2}{\sqrt{y}}$$

$$= -q \; \frac{a}{2} (b^2-a^2) \frac{1}{ab} \left[\frac{1}{b-a} - \frac{1}{b+a} \right]$$

$$= -q \; \frac{\cancel{a}}{\cancel{2}} \cancel{(b^2-a^2)} \frac{1}{\cancel{a}b} \frac{\cancel{2}a}{\cancel{(b^2-a^2)}}$$

$$= -q \, \frac{a}{b}$$

$$= q_i$$

7 We can ask, whether the total induced charge always equals the image charge? No.

⑧ Intuitively, as suggested by Miller (2015 Fall), the field lines for a charge outside the sphere can end at infinity, thus the total induced charge is less than the original charge.

If the charge is inside the sphere the field lines has to end on the surface of the sphere. Thus, the total induced charge in this case is equal to the original charge. But the image charge is greater in magnitude.

⑨ Analyse the case when the charge is placed at the center of the sphere. That is, take the limit $b \to 0$.