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Legendre polynomials

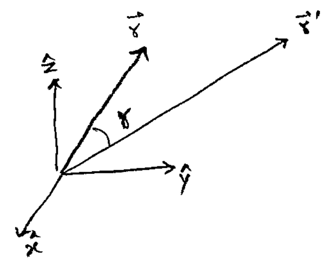
① The electric potential due to a unit point charge in free space, the Green function, is

$$G_0(\vec{r}, \vec{r}\,') = \frac{1}{4\pi\epsilon_0}\,\frac{1}{|\vec{r}-\vec{r}\,'|}$$

② Placing the origin at an arbitrary point in space and using spherical coordinates for $\vec{r}$ and $\vec{r}\,'$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,'}}$$

$$= \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}}$$



where

$$\vec{r}\cdot\vec{r}\,' = rr'\cos\gamma$$
$$= rr'\left[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\right]$$

③ If we choose $\vec{r}\,'$ to be on the z-axis, we have $\theta' = 0$, which implies

$$\cos\gamma = \cos\theta .$$

Thus,

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} \qquad \theta' = 0 .$$

(4) Let us further restrict $\vec{r}$ to be on the z-axis, that is $\theta = 0$, which implies.

$\theta = \theta' = 0$.

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'}}$$

$$= \frac{1}{\sqrt{(r-r')^2}}$$

$$= \begin{cases} \dfrac{1}{r-r'}, & \text{if } r' < r, \\ \dfrac{1}{r'-r}, & \text{if } r < r'. \end{cases}$$

(5) Using the series expansion, for $x < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

we can write

$$\frac{1}{|\vec{r}-\vec{r}'|} = \begin{cases} \dfrac{1}{r}\left[1 + \dfrac{r'}{r} + \left(\dfrac{r'}{r}\right)^2 + \cdots\right], & \text{for } r' < r, \\ \dfrac{1}{r'}\left[1 + \dfrac{r}{r'} + \left(\dfrac{r}{r'}\right)^2 + \cdots\right], & \text{for } r < r' \end{cases}$$

$$= \frac{1}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>}\right)^l \qquad \text{for } \theta = \theta' = 0.$$

where

$$r_< = \text{Minimum}(r, r')$$

$$r_> = \text{Maximum}(r, r')$$

6 Let us now remove the restriction on θ, keeping $\theta' = 0$. Thus,

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} \qquad \theta' = 0 .$$

The coefficients in the series expansion in this case will be defined as the Legendre polynomials:

$$\frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} = \frac{1}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>}\right)^l P_l(\cos\theta) .$$

7 Let us determine $P_l(\cos\theta)$ for $l = 0, 1, 2, 3$ using Taylor expansion, $r > r'$,

$$\frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} = \frac{1}{r} \frac{1}{\left(1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta\right)^{\frac{1}{2}}}$$

$$= \frac{1}{r} \frac{1}{\sqrt{1 - \left\{2\frac{r'}{r}\cos\theta - \left(\frac{r'}{r}\right)^2\right\}}}$$

$$= \frac{1}{r}\Big[1 + \frac{1}{2}\left\{2\frac{r'}{r}\cos\theta - \left(\frac{r'}{r}\right)^2\right\} + \frac{1}{2!}\frac{3}{4}\left\{2\frac{r'}{r}\cos\theta - \left(\frac{r'}{r}\right)^2\right\}^2 + \frac{1}{3!}\frac{3 \cdot 5}{2^3}\left\{2\frac{r'}{r}\cos\theta - \left(\frac{r'}{r}\right)^2\right\}^3 \cdots \Big]$$

(8)
$$\frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} = \frac{1}{r}\Bigg[1 + \frac{1}{2}\left\{2\,\frac{r'}{r}\cos\theta - \left(\frac{r'}{r}\right)^2\right\}$$
$$+ \frac{1}{2!}\frac{3}{2^2}\left\{4\left(\frac{r'}{r}\right)^2\cos^2\theta - 4\left(\frac{r'}{r}\right)^3\cos\theta + \left(\frac{r'}{r}\right)^4\right\}$$
$$+ \frac{1}{3!}\frac{3\times5}{2^3}\left\{\left(\frac{r'}{r}\right)^3 2^3\cos^3\theta + O\left(\frac{r'}{r}\right)^4\right\}$$
$$\cdots \Bigg]$$

$$= \frac{1}{r}\Bigg[1 + \frac{r'}{r}\left\{\cos\theta\right\}$$
$$+ \left(\frac{r'}{r}\right)^2\left\{-\frac{1}{2} + \frac{3}{2}\cos^2\theta\right\}$$
$$+ \left(\frac{r'}{r}\right)^3\left\{-\frac{3}{2}\cos\theta + \frac{5}{2}\cos^3\theta\right\}\Bigg]$$
$$+ \cdots$$

(9) Thus we identify

$l=0$ $\quad P_0(\cos\theta) = 1$

$l=1$ $\quad P_1(\cos\theta) = \cos\theta$

$l=2$ $\quad P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$

$l=3$ $\quad P_3(\cos\theta) = \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$

(10) In general

$$P_l(\cos\theta) = \left(\frac{d}{d\cos\theta}\right)^l \frac{(\cos^2\theta - 1)^l}{2^l\, l!}$$

or

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l\, l!}$$