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Multipole expansion

① Taylor expansion of a function leads to the following observation

$$f(x+a) = f(x) + a\frac{d}{dx}f(x) + \frac{1}{2!}a^2\frac{d^2}{dx^2}f(x) + \cdots$$

$$= \left[1 + \left(a\frac{d}{dx}\right) + \frac{1}{2!}\left(a\frac{d}{dx}\right)^2 + \cdots\right] f(x)$$

$$= \sum_{l=0}^{\infty} \frac{1}{l!}\left(a\frac{d}{dx}\right)^l f(x)$$

$$= e^{a\frac{d}{dx}} f(x)$$

② We can extend this to three dimensional space,

$$f(\vec{r}+\vec{a}) = e^{\vec{a}\cdot\vec{\nabla}} f(\vec{r}).$$

③ In particular we have the free Green's function

$$G_0(\vec{r}-\vec{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0} e^{-\vec{r}'\cdot\vec{\nabla}} \frac{1}{|\vec{r}|}$$

$$= \frac{1}{4\pi\epsilon_0} e^{-\vec{r}'\cdot\vec{\nabla}} \frac{1}{r}$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{l!}\left(-\vec{r}'\cdot\vec{\nabla}\right)^l \frac{1}{r}$$

4 $\underline{\underline{l=0}}$:

$$\frac{1}{l!}\left(-\vec{r}'\cdot\vec{\nabla}\right)^l \frac{1}{r} = \frac{1}{r}$$

$\underline{\underline{l=1}}$:

$$\frac{1}{l!}\left(-\vec{r}'\cdot\vec{\nabla}\right)^l \frac{1}{r} = -\vec{r}'\cdot\vec{\nabla}\frac{1}{r}$$

$$= \frac{\vec{r}'\cdot\vec{r}}{r^3}$$

5 Thus,

$$G_0(\vec{r}-\vec{r}') = \underbrace{\frac{1}{4\pi\epsilon_0}\frac{1}{r}}_{l=0} + \underbrace{\frac{1}{4\pi\epsilon_0}\frac{\vec{r}\cdot\vec{r}'}{r^3}}_{l=1} + \cdots$$

6 The electric potential for an arbitrary charge distribution $\rho(\vec{r}')$ is

$$\phi(\vec{r}) = \int d^3r'\, G_0(\vec{r},\vec{r}')\,\rho(\vec{r}')$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\underbrace{\boxed{\int d^3r'\,\rho(\vec{r}')}}_{\substack{l=0\\ \text{total charge}}} + \frac{1}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}\cdot\underbrace{\boxed{\int d^3r'\,\vec{r}'\,\rho(\vec{r}')}}_{\substack{l=1\\ \text{dipole moment.}}} + \cdots$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\vec{r}\cdot\vec{d}}{r^3} + \cdots\right]$$

where $Q = \int d^3r'\,\rho(\vec{r}')$

$\vec{d} = \int d^3r'\,\vec{r}'\,\rho(\vec{r}')$.