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An integral involving Legendre polynomial

① Consider the integral

$r^2 + r'^2 - 2rr'\cos\theta =$

$$I(r,r') = \int_0^\pi \sin\theta\, d\theta \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}$$

$$= \frac{1}{2rr'} \int_{(r-r')^2}^{(r+r')^2} \frac{dy}{\sqrt{y}}$$

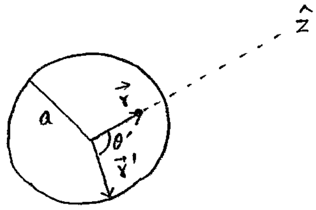
$$= \frac{1}{rr'} \sqrt{y} \Big|_{(r-r')^2}^{(r+r')^2}$$

$$= \begin{cases} \frac{2}{r}, & \text{if } r' < r, \\ \frac{2}{r'}, & \text{if } r < r'. \end{cases}$$

$$= \frac{2}{r_>}$$

② This integral is the primary reason for the difference in the amount of induced charge on a spherical conducting shell for the cases when the charge is inside versus outside. (Refer solutions to Midterm - 01.)

3 This integral is also the reason why electric field is zero inside a uniformly charged spherical shell.



The charge density for a uniformly charged spherical shell is

$$\rho(\vec{r}') = \frac{Q}{4\pi a^2}\,\delta(r'-a)$$

4 The electric potential is given using the free Green's function

$$\phi(\vec{r}) = \int d^3r'\; G_0(\vec{r},\vec{r}')\,\rho(\vec{r}')$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^\infty r'^2 dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' \;\frac{1}{|\vec{r}-\vec{r}'|}\;\frac{Q}{4\pi a^2}\,\delta(r'-a)$$

choose z-axis along $\vec{r}$ here.

$$= \frac{Q}{4\pi\epsilon_0}\,\frac{1}{4\pi a^2}\int_0^\infty r'^2 dr' \int_0^\pi \sin\theta' d\theta' \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} \;\frac{\delta(r'-a)}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}$$

$$= \frac{Q}{4\pi\epsilon_0}\,\frac{1}{2}\int_0^\pi \sin\theta' d\theta' \;\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta'}}$$

5 Using the integral in 1 we have the electric potential due to a uniformly charged sphere to be

$$\phi(\vec{r}) = \begin{cases} \dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{a}, & \text{if } r<a, \\ \dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{r}; & \text{if } a<r. \end{cases}$$

6 The electric field is then

$$\vec{\nabla}\frac{1}{r} = -\frac{\hat{r}}{r^2}$$

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$$

$$= \begin{cases} 0, & \text{if } r<a, \\ \dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{r^2}\hat{r}, & \text{if } a<r. \end{cases}$$