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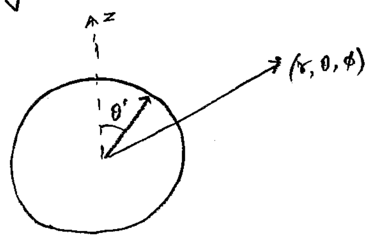
① Consider a spherical shell of radius a with charge distribution in the polar angle alone,

$$\rho(r', \theta', \phi') = \sigma(\theta')\, \delta(r'-a)$$



② The electric potential due to this charge distribution is

$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

$$\phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^\infty r'^2 dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' \frac{\sigma(\theta')\,\delta(r'-a)}{\sqrt{r^2+r'^2-2rr'\cos\gamma}}$$

$$= \frac{2\pi a^2}{4\pi\epsilon_0}\int_0^\pi \sin\theta' d\theta' \frac{\sigma(\theta')}{\sqrt{r^2+a^2-2ar\cos\gamma}}$$

③ On the z-axis, $\theta' = 0$, we have

$$\phi(r,0,0) = \frac{2\pi a^2}{4\pi\epsilon_0}\int_0^\pi \sin\theta' d\theta' \frac{\sigma(\theta')}{\sqrt{r^2+a^2-2ar\cos\theta'}}$$

④ Let us consider the polar charge distribution ($2l$-pole)

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_l(\cos\theta)$$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

$\vdots$

$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l\, l!}$

5 Thus, the potential due to a 2^l-pole, on the z-axis, is

$$\phi(r,0,0) = \frac{Q}{4\pi\epsilon_0}\,\frac{1}{2}\int_0^{\pi} \sin\theta'\,d\theta'\,\frac{P_l(\cos\theta')}{\sqrt{r^2+a^2-2ar\cos\theta'}}$$

6 For $l=0$, using earlier notes, we have.

$$\frac{1}{2}\int_0^{\pi}\sin\theta'\,d\theta'\,\frac{\overbrace{P_0(\cos\theta')}^{=1}}{\sqrt{r^2+a^2-2ar\cos\theta'}} = \begin{cases}\frac{1}{a}, & r<a,\\ \frac{1}{r}, & a<r\end{cases}$$

$$= \frac{1}{r_>}.$$

7 For $l=1$, we have

$\cos\theta' = t$

$-\sin\theta' d\theta' = dt$

$r^2+a^2-2art = y$

$t = \frac{r^2+a^2-y}{2ar}$

$dt = -\frac{dy}{2ar}$

$$\frac{1}{2}\int_0^{\pi}\sin\theta'\,d\theta'\,\frac{\overbrace{P_1(\cos\theta')}^{\cos\theta'}}{\sqrt{r^2+a^2-2ar\cos\theta'}} = \frac{1}{2}\int_{-1}^{+1} dt\,\frac{t}{\sqrt{r^2+a^2-2art}}$$

$$= \frac{1}{2}\int_{(r-a)^2}^{(r+a)^2}\frac{dy}{2ar}\,\frac{(r^2+a^2-y)}{2ar\sqrt{y}}$$

$$= \frac{1}{8a^2r^2}\int_{(r-a)^2}^{(r+a)^2}\frac{dy}{\sqrt{y}}\,(r^2+a^2-y)$$

$$= \frac{1}{8a^2r^2}\left[(r^2+a^2)\int_{(r-a)^2}^{(r+a)^2}\frac{dy}{\sqrt{y}} - \int_{(r-a)^2}^{(r+a)^2} dy\,\sqrt{y}\right]$$

$$= \frac{1}{8a^2r^2}\left[(r^2+a^2)\,2\sqrt{y}\,\Big|_{(r-a)^2}^{(r+a)^2} - \frac{2}{3}y^{\frac{3}{2}}\Big|_{(r-a)^2}^{(r+a)^2}\right]$$

8 Thus,

$$\frac{1}{2}\int_0^{\pi} \sin\theta'\,d\theta' \,\frac{P_1(\cos\theta')}{\sqrt{r^2+a^2-2ar\cos\theta'}} = \frac{1}{8a^2r^2}\Big[(r^2+a^2)\,2\left\{(r+a)-|r-a|\right\} - \frac{2}{3}\left\{(r+a)^3-|r-a|^3\right\}\Big]$$

$$= \begin{cases} \dfrac{1}{8a^2r^2}\Big[2(r^2+a^2)\left\{r+a-(a-r)\right\} - \dfrac{2}{3}\left\{(r+a)^3-(a-r)^3\right\}\Big], & r<a, \\[2ex] \dfrac{1}{8a^2r^2}\Big[2(r^2+a^2)\left\{r+a-(r-a)\right\} - \dfrac{2}{3}\left\{(r+a)^3-(r-a)^3\right\}\Big], & a<r. \end{cases}$$

$$= \begin{cases} \dfrac{1}{8a^2r^2}\Big[4r(r^2+a^2) - \dfrac{2}{3}\Big\{(a^3+3a^2r+3ar^2+r^3) - (a^3-3a^2r+3ar^2-r^3)\Big\}\Big], & r<a, \\[2ex] \dfrac{1}{8a^2r^2}\Big[4a(r^2+a^2) - \dfrac{2}{3}\Big\{(r^3+3r^2a+3ra^2+a^3) - (r^3-3r^2a+3ra^2-a^3)\Big\}\Big], & a<r. \end{cases}$$

$$= \begin{cases} \dfrac{1}{8a^2r^2}\Big[4r^3 + \cancel{4a^2r} - \cancel{4a^2r} - \dfrac{4}{3}r^3\Big], & r<a, \\[2ex] \dfrac{1}{8a^2r^2}\Big[4a^3 + 4r^2a - 4r^2a - \dfrac{4}{3}a^3\Big], & a<r. \end{cases}$$

$$= \begin{cases} \dfrac{1}{3}\,\dfrac{r}{a^2}, & r<a, \\[2ex] \dfrac{1}{3}\,\dfrac{a}{r^2}, & a<r. \end{cases}$$

$$= \frac{1}{3}\,\frac{1}{r_>}\left(\frac{r_<}{r_>}\right).$$

(9) As a homework exercise, you will show,

$$\frac{1}{2}\int_0^{\pi} \sin\theta' d\theta' \frac{P_2(\cos\theta')}{\sqrt{r^2+a^2-2ar\cos\theta'}} = \frac{1}{5}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^2 .$$

(10) In general, we have

$$\frac{1}{2}\int_0^{\pi} \sin\theta' d\theta' \frac{P_l(\cos\theta')}{\sqrt{r^2+a^2-2ar\cos\theta'}} = \frac{1}{(2l+1)}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l .$$

(11) Using (10) in (5) we have the electric potential to be due to $2l$-pole (on the z-axis)

$$\phi(r,0,0) = \frac{Q}{4\pi\epsilon_0}\frac{1}{(2l+1)}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l .$$