Date: 2015 Feb 27

(1) As part of homework No. 3 we discussed the electric potential due to a charge distribution given in terms of Legendre polynomials,

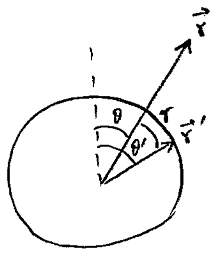
$$\rho(\vec{r}') = \frac{Q}{4\pi a^2} P_l(\cos\theta')\, \delta(r'-a).$$

In this calculation, to be able to use the elementary integrals, we had to choose the observation point on the z-axis. Here we shall redo this calculation without making this choice.

(2) The electric potential is given by

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{4\pi} \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \frac{P_l(\cos\theta')}{\sqrt{r^2+a^2-2ar\cos\gamma}}$$



where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

3. It is clear how choosing $\theta = 0$ leads to

$$\cos\gamma \rightarrow \cos\theta',$$

which simplifies the integral. It is often the case that other restrictions in the problem do not allow us to make this choice. Here we shall find the electric potential using spherical harmonics that does not rely on this restriction.

4. Using

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right)^l \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi)\, Y_{lm}^*(\theta',\phi')$$

where $r_< = \text{Min}(r, r')$ and $r_> = \text{Max}(r, r')$ we can write

$$\frac{1}{\sqrt{r^2+a^2-2ar\cos\gamma}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right)^l \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi)\, Y_{lm}^*(\theta',\phi')$$

where now $r_< = \text{Min}(r, a)$ and $r_> = \text{Max}(r, a)$.

(5) Using (4) in (2) we have

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{4\pi}\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \; P_l(\cos\theta') \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^{l'} \frac{4\pi}{2l'+1} \; Y_{l'm'}(\theta,\phi) \; Y^*_{l'm'}(\theta',\phi')$$

(6) Using

$$P_l(\cos\theta') = \sqrt{\frac{4\pi}{2l+1}} \; Y_{l0}(\theta',\phi') \qquad d\Omega' = d\phi' \sin\theta' d\theta'$$

we have.

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{4\pi}\int d\Omega' \sqrt{\frac{4\pi}{2l+1}} \; Y_{l0}(\theta',\phi') \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^{l'} \frac{4\pi}{2l'+1} \; Y_{l'm'}(\theta,\phi) \; Y^*_{l'm'}(\theta',\phi')$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{4\pi}\sqrt{\frac{4\pi}{2l+1}} \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^{l'} \frac{4\pi}{2l'+1} \; Y_{l'm'}(\theta,\phi) \underbrace{\int d\Omega' \; Y_{l0}(\theta',\phi') \; Y^*_{l'm'}(\theta',\phi')}_{\delta_{ll'}\,\delta_{m'0}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{\cancel{4\pi}}\sqrt{\frac{4\pi}{2l+1}} \; \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^{l} \frac{\cancel{4\pi}}{2l+1} \; Y_{l0}(\theta,\phi)$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{r_>}\frac{1}{2l+1}\left(\frac{r_<}{r_>}\right)^{l} P_l(\cos\theta) .$$