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1. Maxwell's equations

$$\vec{\nabla} \cdot \epsilon_0 \vec{E} = \rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \frac{\vec{B}}{\mu_0} = \frac{\partial}{\partial t} \epsilon_0 \vec{E} + \vec{J}$$

which also imply the current conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

2. Statics involves

$$\frac{\partial \vec{E}}{\partial t} = 0 , \quad \frac{\partial \vec{B}}{\partial t} = 0 , \quad \frac{\partial \rho}{\partial t} = 0 , \quad \frac{\partial \vec{J}}{\partial t} = 0 .$$

For consistency we also need

$$\cancel{\frac{\partial \rho}{\partial t}} + \vec{\nabla} \cdot \vec{J} = 0$$

3. Electrostatics :

$$\vec{\nabla} \cdot \epsilon_0 \vec{E} = \rho$$
$$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla} \phi$$

Magnetostatics :

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

vector potential

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{J} = 0$$

(4) Combining the two equations of magnetic field

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{j}$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{j}$$

(5) Gauge transformations:

→ Maxwell's equations are invariant under

$$\vec{A} \to \vec{A} + \vec{\nabla}\lambda, \qquad \lambda = \lambda(\vec{r}, t) \text{ arbitrary}$$

$$\phi \to \phi - \frac{\partial}{\partial t}\lambda.$$

That is,

$$\vec{E} \to -\vec{\nabla}\left(\phi - \frac{\partial \lambda}{\partial t}\right) - \frac{\partial}{\partial t}\left(\vec{A} + \vec{\nabla}\lambda\right) = \vec{E}$$

$$\vec{B} \to \vec{\nabla} \times (\vec{A} + \vec{\nabla}\lambda) = \vec{B}$$

→ This allows us to work in a gauge. In magnetostatics we shall work is the radiation (Coulomb) gauge:

$$\vec{\nabla} \cdot \vec{A} = 0.$$

(6) Using the radiation gauge of (5) in (4) we have

$$-\nabla^2 \vec{A} = \mu_0 \vec{j} .$$

We can write $\vec{A}$ using the Green's function

$$-\nabla^2 G(\vec{r},\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,') \qquad G(\vec{r},\vec{r}\,') = \frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}\,'|}$$

to obtain

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} .$$

(7) The magnetic field is determined using

$$\begin{aligned}
\vec{B}(\vec{r}) &= \vec{\nabla}\times\vec{A} \\
&= \vec{\nabla}\times \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \\
&= \frac{\mu_0}{4\pi}\int d^3r' \left(\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}\,'|}\right) \times \vec{j}(\vec{r}\,') \\
&= -\frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \times \vec{j}(\vec{r}\,') \\
&= \frac{\mu_0}{4\pi}\int d^3r' \, \vec{j}(\vec{r}\,') \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}
\end{aligned}$$

which is the statement of Biot - Savart law.