Meissner effect (Ref. London and London, Proc. R. Soc., 149 (1935) 71.

(1) Conductivity in a typical conductors (like copper and gold) is described by the Ohm's law

$$\vec{j} = \sigma \vec{E},$$

where σ is the static conductivity. Conductivity in superconductors is described by London's "acceleration equation"

$$\mu_0 \frac{d\vec{j}}{dt} = \frac{1}{\lambda_L^2} \vec{E},$$

where λ_L is the London penetration depth.

(2) To model the above equations, let us describe the electrons in the material using Newton's law

$$m \frac{d^2\vec{x}}{dt^2} + m\Gamma \frac{d\vec{x}}{dt} + m\omega_0^2 \vec{x} = e\vec{E}$$

inertia term in Newton's law

damping term depending on velocity

binds the electron to the atom

force due to externally applied electric field

3. Ohm's law is obtained by presuming that on the left hand side, the damping term dominates

$$m\gamma\vec{v} = e\vec{E}$$

$$\Rightarrow \quad \vec{v} = \frac{e}{m\gamma}\vec{E}.$$

4. Using the above in the definition of current we obtain the Ohm's law $n = \frac{\text{number of charges}}{\text{volume}}$

$$\vec{j} = ne\vec{v}$$

$$= \frac{ne^2}{m\gamma}\vec{E} \qquad \sigma = \frac{ne^2}{m\gamma}$$

$$= \sigma\vec{E}$$

5. London's "acceleration equation" is obtained by presuming that on the left hand side, the inertia term dominates

$$m\frac{d\vec{v}}{dt} = e\vec{E}$$

$$\Rightarrow \quad \frac{d\vec{v}}{dt} = \frac{e}{m}\vec{E}.$$

6 Thus, combing with $\vec{j} = ne\vec{v}$ we obtain

$$\frac{d\vec{j}}{dt} = ne\frac{d\vec{v}}{dt}$$

$$= \frac{ne^2}{m}\vec{E} \qquad (\text{using } 5)$$

$$= \frac{1}{\mu_0 \lambda_L^2}\vec{E} \qquad \lambda_L^2 = \frac{m}{\mu_0 ne^2} = \frac{m\epsilon_0 c^2}{ne^2}$$

Notice, how steady currents are possible for zero electric field!

7 Taking the curl of the equation in 6 we obtain, (ignoring the difference between $\frac{\partial}{\partial t}$ and $\frac{d}{dt}$,)

$$\frac{\partial}{\partial t}\vec{\nabla}\times\vec{j} = \frac{1}{\mu_0\lambda_L^2}\vec{\nabla}\times\vec{E}$$

$$= -\frac{1}{\mu_0\lambda_L^2}\frac{\partial\vec{B}}{\partial t} \qquad \left(\text{using } \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\right)$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{j} + \frac{1}{\mu_0\lambda_L^2}\vec{B}\right) = 0$$

The Londons postulated that

$$\vec{\nabla}\times\vec{j} + \frac{1}{\mu_0\lambda_L^2}\vec{B} = 0,$$

which is consistant. Notice, in general, any constant vector (in time) would have been sufficient.

8 Thus, the following two postulates describe a superconductor :

Postulate (i)

$$\mu_0 \lambda_L^2 \frac{\partial \vec{j}}{\partial t} = \vec{E}$$

Postulate (ii)

$$-\mu_0 \lambda_L^2 \vec{\nabla} \times \vec{j} = \vec{B}$$

9 A dramatic experimental observation involving superconductors is that is expels the magnetic field in such a material, below a critical temperature. This is the Meissner effect. We shall next show that Londons postulate together with the Maxwell equations predict Meissner effect.

10 Let us begin with Maxwell's equation, and take curl,

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{E} + \mu_0 \vec{j}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{\nabla} \times \vec{E} + \mu_0 \vec{\nabla} \times \vec{j}$$

$$\vec{\nabla} \underbrace{(\vec{\nabla} \cdot \vec{B})}_{=0} - \nabla^2 \vec{B} = -\underbrace{\mu_0 \epsilon_0}_{\frac{1}{c^2}} \frac{\partial}{\partial t} \frac{\partial \vec{B}}{\partial t} - \frac{1}{\lambda_L^2} \vec{B}$$

$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}\, \vec{\nabla} \cdot \vec{B} - \nabla^2 \vec{B}$

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$-\mu_0 \lambda_L^2 \vec{\nabla} \times \vec{j} = \vec{B}$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{B} = \frac{1}{\lambda_L^2} \vec{B}$$

(11) For static case, setting $\frac{\partial}{\partial t}\vec{B} = 0$, we have

$$\nabla^2 \vec{B} = \frac{1}{\lambda_L^2}\vec{B},$$

which for planar geometry implies

$$\vec{B} = \vec{B}_0\, e^{-\frac{x}{\lambda_L}},$$

where λ_L is interpreted as the Londons penetration depth of magnetic field inside a superconductor. For gold (electron number density $n \sim 6\times10^{28}\,\frac{1}{m^3}$) we have

$$\lambda_L^2 = \frac{m}{\mu_0 n e^2}$$

$$= \frac{9.1\times10^{-31}\,kg}{\left(4\pi\times10^{-7}\,\frac{N s^2}{C^2}\right)\times\left(6\times10^{28}\,\frac{1}{m^3}\right)\times\left(1.6\times10^{-19}\,C\right)^2}$$

$$= 4.7\times10^{-16}\,m^2$$

That is,

$$\lambda_L \sim 10^{-8}\,m = 10\,nm.$$