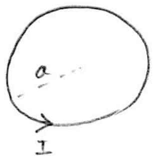
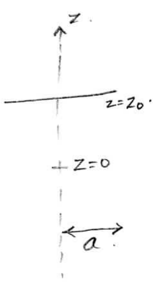
Date: 2015 Mar 23

Circular current carrying wire. ($\vec{B}$ on axis)

① The current density for a steady current flowing in a circular loop of wire is

$$\vec{j}(\vec{r}) = \hat{\phi}\, I\, \delta(\rho - a)\, \delta(z - z_0)$$





$$\hat{\phi}' = -\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j}$$

② $$\vec{A}(\overset{z,\rho,\phi}{\overset{\uparrow}{\vec{r}}}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{\mu_0}{4\pi}\int_{-\infty}^{+\infty} dz' \int_0^{\infty} \rho' d\rho' \int_0^{2\pi} d\phi' \, \frac{I\,\hat{\phi}'\,\delta(\rho'-a)\,\delta(z'-z_0)}{\sqrt{(z-z')^2 + \rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi')}}$$

$$= \frac{\mu_0 I}{4\pi}\, a \int_0^{2\pi} d\phi' \, \frac{[-\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j}]}{\sqrt{(z-z_0)^2 + \rho^2 + a^2 - 2\rho a\cos(\phi-\phi')}}$$

③ On the symmetry axis of the loop we have

$$\vec{A}(z, \rho=0, \phi=0) = \frac{\mu_0 I}{4\pi}\, a \int_0^{2\pi} d\phi' \, \frac{[-\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j}]}{\sqrt{(z-z_0)^2 + a^2}}$$

$$= 0$$

This is also true for the case $\rho << a$ and $\rho << |z-z_0|$.

④ Does this imply

$$\vec{B} = \vec{\nabla} \times \vec{A} = 0 \ ?$$

No. We need $\vec{A}$ as a function of ρ and ϕ to conclude about $\vec{B}$.

⑤ Thus, we begin again, now using

$$\vec{B}(\overset{z,\rho,\phi}{\vec{r}}) = \frac{\mu_0}{4\pi} \int d^3r' \ \vec{j}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

$$= \frac{\mu_0}{4\pi} \int_{-\infty}^{+\infty} dz' \int_0^\infty \rho' d\rho' \int_0^{2\pi} d\phi' \ I \ \hat{\phi}' \ \delta(z'-z_0)\, \delta(\rho'-a) \times \frac{\left[\overbrace{(x-x')\hat{i} + (y-y')\hat{j}}^{\vec{\rho} - \vec{\rho}'} + (z-z')\hat{k} \right]}{\left[(z-z')^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi-\phi') \right]^{\frac{3}{2}}}$$

$$= \frac{\mu_0 I}{4\pi} \ a \int_0^{2\pi} d\phi' \ \frac{\hat{\phi}' \times \left[\vec{\rho} - a\hat{r}' + (z-z_0)\hat{k} \right]}{\left[(z-z_0)^2 + \rho^2 + a^2 - 2\rho a \cos(\phi-\phi') \right]^{\frac{3}{2}}}$$

⑥ On the symmetry axis we have

$$\vec{B}(z, \rho=0, \phi=0) = \frac{\mu_0 I}{4\pi} \ a \int_0^{2\pi} d\phi' \ \frac{\hat{\phi}' \times \left[-a\hat{r}' + (z-z_0)\hat{k} \right]}{\left[(z-z_0)^2 + a^2 \right]^{3/2}}.$$

$$= \frac{\mu_0 I}{4\pi} \ \frac{a}{\left[(z-z_0)^2 + a^2 \right]^{\frac{3}{2}}} \int_0^{2\pi} d\phi' \left[a\hat{z} + \underbrace{(z-z_0)\hat{r}'}_{\text{integrates to zero.}} \right]$$

$$= \frac{\mu_0 I}{4\pi} \ \frac{2\pi a^2}{\left[(z-z_0)^2 + a^2 \right]^{\frac{3}{2}}} \xrightarrow{a \ll |z-z_0|} \frac{\mu_0 I}{4\pi} \ \frac{2\pi a^2}{(z-z_0)^3}$$

$$\xrightarrow{|z-z_0| \ll a.} \frac{\mu_0 I}{2a}$$