Date : 2015 Mar 25

Circular current carrying wire ($\vec{B}$ in far field)

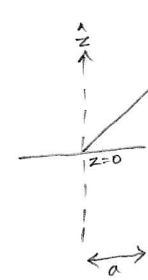
1. We calculated the magnetic field on the symmetry axis of a circular current carrying wire. Here we will calculate the magnetic field in regions where

$$a << r,$$

the so-called far-field.

2. The current density for the circular loop is

$$\vec{j}(\vec{r}) = \hat{\phi}\, I\, \delta(\rho - a)\, \delta(z)$$



3.
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$= \frac{\mu_0}{4\pi} \int_{-\infty}^{+\infty} dz' \int_0^{\infty} \rho' d\rho' \int_0^{2\pi} d\phi' \frac{I\, \hat{\phi}'\, \delta(\rho' - a)\, \delta(z')}{\sqrt{(z-z')^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')}}$$

$$= \frac{\mu_0 I}{4\pi}\, a \int_0^{2\pi} d\phi' \frac{[-\sin\phi'\, \hat{i} + \cos\phi'\, \hat{j}]}{\sqrt{\underbrace{z^2 + \rho^2}_{r^2} + a^2 - 2\rho a \cos(\phi - \phi')}}$$

(4) In the far-field approximation,

$$a << r,$$

we have

$$\frac{1}{\sqrt{r^2 + a^2 - 2\rho a \cos(\phi-\phi')}} = \frac{1}{r} \frac{1}{\sqrt{1 + \frac{a^2}{r^2} - 2\frac{\rho a}{r^2}\cos(\phi-\phi')}}$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\frac{a^2}{r^2} + \frac{1}{2} 2 \frac{\rho a}{r^2}\cos(\phi-\phi') + \text{higher orders}\right]$$

using

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \cdots \quad \text{for} \quad x < 1.$$

(5) Using (4) in (3)

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\frac{a}{r}\int_0^{2\pi} d\phi' \left[-\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j}\right]\Big[\underbrace{1 - \frac{1}{2}\frac{a^2}{r^2}}_{=0 \text{ after } \phi' \text{ integrals.}} + \frac{\rho a}{r^2}\cos(\phi-\phi') + \cdots\Big]$$

$$= \frac{\mu_0 I}{4\pi}\frac{a^2\rho}{r^3}\int_0^{2\pi} d\phi' \left[-\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j}\right]\cos(\phi-\phi')$$

(6)

$$\int_0^{2\pi} d\phi' \sin\phi' \cos(\phi-\phi') = \cos\phi \underbrace{\int_0^{2\pi} d\phi' \sin\phi' \cos\phi'}_{=0} + \sin\phi \underbrace{\int_0^{2\pi} d\phi' \sin^2\phi'}_{=\pi}$$

$$= \pi \sin\phi$$

$$\int_0^{2\pi} d\phi' \cos\phi' \cos(\phi-\phi') = \cos\phi \underbrace{\int_0^{2\pi} d\phi' \cos^2\phi'}_{=\pi} + \sin\phi \underbrace{\int_0^{2\pi} d\phi' \cos\phi' \sin\phi'}_{=0}$$

$$= \pi \cos\phi$$

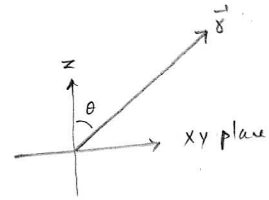
7 Thus we have

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{(I \pi a^2)\, \rho}{r^3} \left[-\sin\phi\, \hat{i} + \cos\phi\, \hat{j} \right]$$

$$= \frac{\mu_0}{4\pi} \frac{(I \pi a^2)\, \rho}{r^3}\, \hat{\phi}$$

8 Let us define the magnetic dipole moment for a current carrying loop as

$\vec{m} = I \text{ (Area) } \hat{n}$ → direction perpendicular to area and obtained (using right hand rule) when curl as the direct of thumb represents the current.



Thus,

$$(I \pi a^2)\, \rho\, \hat{\phi} = m \rho \hat{\phi}$$

$$= m r \sin\theta\, \hat{\phi} \qquad \rho = r\sin\theta$$

$$= m \hat{z} \times \vec{r} \qquad \hat{z} \times \hat{r} = \hat{\phi}$$

$$= \vec{m} \times \vec{r}$$

9 Thus, the vector potential for a magnetic dipole moment is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$

(10) Magnetic field is derived as

$$\vec{\nabla}\cdot\frac{\vec{r}}{r^3} = 4\pi\,\delta^{(3)}(\vec{r})$$

$$\vec{\nabla}\,\frac{\vec{r}}{r^3} = \frac{\overleftrightarrow{I} - 3\,\hat{r}\,\hat{r}}{r^3}$$

$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}$$

$$= \frac{\mu_0}{4\pi}\,\vec{\nabla}\times\left(\vec{m}\times\frac{\vec{r}}{r^3}\right)$$

$$= \frac{\mu_0}{4\pi}\,\vec{m}\;\vec{\nabla}\cdot\frac{\vec{r}}{r^3} - \frac{\mu_0}{4\pi}\,\vec{m}\cdot\vec{\nabla}\,\frac{\vec{r}}{r^3}$$

$$= \mu_0\,\vec{m}\;\delta^{(3)}(\vec{r}) + \frac{\mu_0}{4\pi}\,\frac{\vec{m} - 3\,\vec{m}\cdot\hat{r}\,\hat{r}}{r^3}$$

(11) The δ-function in the magnetic field is necessary to satisfy

$$\vec{\nabla}\cdot\vec{B} = \mu_0\,\vec{m}\cdot\vec{\nabla}\,\delta^{(3)}(\vec{r}) - \frac{\mu_0}{4\pi}\,\vec{m}\cdot\vec{\nabla}\,\underbrace{\vec{\nabla}\cdot\frac{\vec{r}}{r^3}}_{4\pi\,\delta^{(3)}(\vec{r})}$$

$$= \mu_0\,\vec{m}\cdot\vec{\nabla}\,\delta^{(3)}(\vec{r}) - \mu_0\,\vec{m}\cdot\vec{\nabla}\,\delta^{(3)}(\vec{r})$$

$$= 0.$$

(12) Check: $\vec{\nabla}\cdot\vec{A} = 0$ (homework)