Date: 2015 Mar 27

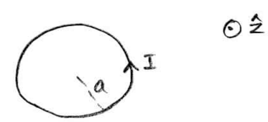
Magnetic dipole moment (Multipole expansion)

1. We found the vector potential for a circular current carrying wire to be

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$

and the magnetic field to be

$$\vec{B}(\vec{r}) = \mu_0 \vec{m}\, \delta^{(3)}(\vec{r}) + \frac{\mu_0}{4\pi} \frac{3\, \vec{m}\cdot\hat{r}\,\hat{r} - \vec{m}}{r^3}$$



where

$$\vec{m} = I A\, \hat{z}.$$

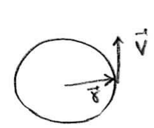
2. In general the magnetic moment is

$$\vec{m} = \frac{q}{2m} \vec{L}$$

$$= \frac{q}{2m} \vec{r} \times m\vec{v}$$

$$= \frac{1}{2} \int d^3r'\, \vec{r}' \times \vec{j}(\vec{r}')$$

For a circular wire we have



$$\vec{m} = \frac{q}{2\cancel{m}} \vec{r} \times \cancel{m}\vec{v}$$

$$= \hat{z}\, \frac{q}{2}\, a\, \frac{2\pi}{T} a$$

$$= \hat{z}\, \frac{q}{T} \pi a^2$$

$$= \hat{z}\, I \pi a^2$$

$$= \hat{z}\, I A.$$

3 Let us consider an arbitrary current density and calculate the vector potential in the far-field approximation — the multipole expansion —

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = e^{-\vec{r}'\cdot\vec{\nabla}} \frac{1}{r} = \frac{1}{r} - \vec{r}'\cdot\vec{\nabla}\frac{1}{r} + \dots = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \dots$$

$$= \frac{\mu_0}{4\pi} \int d^3r'\, \vec{j}(\vec{r}') \left[\frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \dots \right]$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r} \int d^3r'\, \vec{j}(\vec{r}') + \frac{\mu_0}{4\pi} \frac{\vec{r}}{r^3} \cdot \int d^3r'\, \vec{r}'\, \vec{j}(\vec{r}') + \dots$$

4 In magnetostatics we have, using $\vec{\nabla}\times\vec{B} = \mu_0 \vec{j}$,

$$\vec{\nabla}\cdot\vec{j} = 0$$

$$\vec{r}\ \vec{\nabla}\cdot\vec{j} = 0$$

$$\int d^3r\ \vec{r}\ \vec{\nabla}\cdot\vec{j} = 0$$

$$\int d^3r\ \vec{j}\cdot\underbrace{\vec{\nabla}\,\vec{r}}_{\vec{I}} + \underbrace{\text{surface term}}_{=0 \text{ if } \vec{j}=0 \text{ at boundary.}} = 0$$

$$\int d^3r\ \vec{j} = 0$$

Thus, the first term in the multipole expansion of 3 contributes zero.

(5) Repeating the same process for

$$\vec{r}\ \vec{r}\ \vec{\nabla}\cdot\vec{j} = 0$$

we obtain

$$\int d^3r \left[(\vec{j}\cdot\underbrace{\vec{\nabla}\vec{r}}_{\vec{I}})\,\vec{r} + \vec{r}\,(\vec{j}\cdot\underbrace{\vec{\nabla}\vec{r}}_{\vec{I}}) \right] = 0$$

$$\int d^3r \left[\vec{j}\,\vec{r} + \vec{r}\,\vec{j} \right] = 0$$

(6) Thus, the multipole expansion in (3) is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{r}}{r^3}\cdot\int d^3r'\,\vec{r}'\,\vec{j}$$

$$= \frac{\mu_0}{4\pi}\frac{\vec{r}}{r^3}\cdot\underbrace{\int d^3r' \left[\vec{r}'\,\vec{j} + \vec{j}\,\vec{r}' \right.}_{=0} \left. + \vec{r}'\,\vec{j} - \vec{j}\,\vec{r}' \right]\frac{1}{2}$$

$$= \frac{\mu_0}{4\pi}\,\frac{1}{2}\,\frac{1}{r^3}\int d^3r' \left[\vec{r}\cdot\vec{r}'\,\vec{j} - \vec{r}\cdot\vec{j}\,\vec{r}' \right]$$

$$= \frac{\mu_0}{4\pi}\,\frac{1}{2}\int d^3r'\,\frac{\vec{r}\times(\vec{j}\times\vec{r}')}{r^3}$$

$$= \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\vec{r}}{r^3}$$

which is true for an arbitrary current density $\vec{j}(\vec{r})$ in the far-field approximation.