① Refer: The gravity tunnel in a non-uniform Earth
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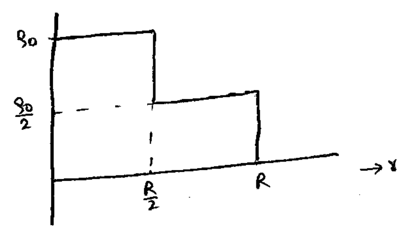
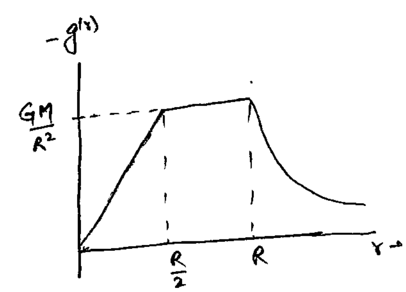
② Approximate Density profile of Earth

$$\rho_0 = \frac{16}{9} \frac{M}{\frac{4\pi}{3} R^3}$$



③ This leads to the gravitational field profile, after approximation in the $\frac{R}{2} < r < R$ region ?

$$g(r) = \begin{cases} -\frac{GM}{R^2}\frac{2r}{R}, & 0 < r < \frac{R}{2}, \\ -\frac{GM}{R^2}, & \frac{R}{2} < r, R, \\ -\frac{GM}{r^2}, & R < r. \end{cases}$$



④

$$\vec{F}(r) = -\vec{\nabla} U$$

$$m\, g(r) = -\frac{\partial}{\partial r} U \qquad V = \frac{U}{m}. \qquad V(\infty) = 0$$

$$g(r) = -\frac{\partial}{\partial r} V$$

$$V(r) = \int dr\, g(r) + C$$

5 Thus,

$$V(r) = \begin{cases} -\frac{7}{4}\frac{GM}{R} + \frac{GM}{R}\left(\frac{r}{R}\right)^2 , & 0 < r < \frac{R}{2} , \\ -2\frac{GM}{R} + \frac{GM}{R}\frac{r}{R} , & \frac{R}{2} < r , R , \\ -\frac{GM}{r} , & R < r . \end{cases}$$

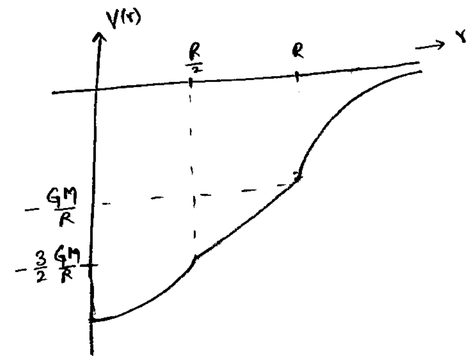
Note:

$$V(0) = -\frac{7}{4}\frac{GM}{R}$$

$$V\left(\frac{R}{2}\right) = -\frac{3}{2}\frac{GM}{R}$$

$$V(R) = -\frac{GM}{R}$$

$$V(\infty) = 0$$



6

$$E = \frac{1}{2}mv^2 + U(r)$$

$$-\frac{GM}{R} = \frac{1}{2}mv^2 + U(r)$$

$$\frac{T}{4} = -\int_R^0 \frac{dr}{\sqrt{2\left[-\frac{GM}{R} - V(r)\right]}}$$

$$= \sqrt{\frac{R^3}{GM}} \int_0^R \frac{dr}{\sqrt{2\left[-R^2 - \frac{R^3}{GM}V(r)\right]}}$$

7 $$\frac{T}{4} = \sqrt{\frac{R^3}{GM}}\left[\int_0^{R/2}\frac{dr}{\sqrt{2\left[-R^2+\frac{7}{4}R^2-r^2\right]}} + \int_{\frac{R}{2}}^{R}\frac{dr}{\sqrt{2\left[-R^2+2R^2-rR\right]}}\right]$$

$$= \sqrt{\frac{R^3}{GM}}\left[\int_0^{\frac{R}{2}}\frac{dr}{\sqrt{\frac{3}{2}R^2-2r^2}} + \int_{\frac{R}{2}}^{R}\frac{dr}{\sqrt{2(R^2-rR)}}\right]$$

$$= \sqrt{\frac{R^3}{GM}}\left[\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{\frac{3}{2}-2x^2}} + \int_{\frac{1}{2}}^{1}\frac{dx}{\sqrt{2(1-x)}}\right]$$

$$= \sqrt{\frac{R^3}{GM}}\left[\frac{1}{\sqrt{2}}\sin^{-1}\frac{1}{\sqrt{3}} + 1\right]$$

$$= \sqrt{\frac{R^3}{GM}}\ \frac{\pi}{2} \times 0.9137$$

$$\pi\sqrt{\frac{R^3}{GM}} \approx 42 \text{ min.}$$

8 $$\frac{T}{2} = \sqrt{\frac{R^3}{GM}}\ \pi \times 0.9137$$

$$= 38.38 \text{ min.}$$