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1. The complete Maxwell equations are

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}$$

2. We can derive

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{B}$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}\right)$$

$$-\left(\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\right) \vec{E} = -\frac{1}{\epsilon_0} \vec{\nabla} \rho - \mu_0 \frac{\partial}{\partial t} \vec{j}$$

and

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{\nabla} \times \vec{E} + \mu_0 \vec{\nabla} \times \vec{j}$$

$$\vec{\nabla}(\underbrace{\vec{\nabla} \cdot \vec{B}}_{=0}) - \nabla^2 \vec{B} = -\mu_0 \epsilon_0 \frac{\partial}{\partial t} \frac{\partial \vec{B}}{\partial t} + \mu_0 \vec{\nabla} \times \vec{j}$$

$$-\left(\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\right) \vec{B} = \mu_0 \vec{\nabla} \times \vec{j}$$

3. If we restrict our analysis to region where $\rho = 0$ and $\vec{j} = 0$ we have

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{E}$$

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{B}$$

4 A wave equation (in 1-D) is

$$\frac{\partial^2}{\partial z^2} f(z,t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} f(z,t),$$

which is satisfied by functional forms

$$f(z \pm vt).$$

The speed of the wave is v.

5 Thus, we can conclude that electric field $\vec{E}$ and magnetic field $\vec{B}$ propagate with speed

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$= \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}} \sim 2.99 \times 10^8 \frac{m}{s}.$$

6 Einstein went ahead and postulated that c is a universal constant. Thus, a spherical wave should look the same for every observer,

$$r'^2 - c^2 t'^2 = r^2 - c^2 t^2$$

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = x^2 + y^2 + z^2 - c^2 t^2$$

7 Let

$$z' = A(v)\, z + B(v)\, ct \qquad x' = x$$

$$ct' = E(v)\, z + F(v)\, ct \qquad y' = y$$

8 So,

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = x^2 + y^2 + (Az + Bct)^2 - (Ez + Fct)^2$$

$$= x^2 + y^2 + (A^2 - E^2)\, z^2 - c^2(F^2 - B^2) + 2zct(AB - EF)$$

Thus, if the spherical wavefront of light look the same to each observer we require

$$A^2 - E^2 = 1$$

$$F^2 - B^2 = 1$$

$$AB = EF$$

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$$A^2 - \left(\frac{AB}{F}\right)^2 = 1$$

$$A^2 - \frac{A^2 B^2}{1 + B^2} = 1$$

$$A^2\left(1 - \frac{B^2}{1+B^2}\right) = 1$$

$$\frac{A^2}{1+B^2} = 1$$

$$A^2 - B^2 = 1$$

Thus, a plausible solution is

$$A = \cosh\theta$$

$$B = \sinh\theta$$

$$F = \cosh\theta$$

$$E = \sinh\theta.$$

(10) Thus, the transformations that will keep the wavefront of light invariant is

$$z' = \cosh\theta \; z + \sinh\theta \; ct$$
$$ct' = \sinh\theta \; z + \cosh\theta \; ct$$

(11) The physical connection is obtained by letting $\beta = \frac{v}{c}$.

$$\tanh\theta = \pm \frac{v}{c}.$$

$$\cosh\theta = \frac{\cosh\theta}{\sqrt{\cosh^2\theta - \sinh^2\theta}} = \frac{1}{\sqrt{1 - \tanh^2\theta}} = \frac{1}{\sqrt{1-\beta^2}} = \gamma$$

$$\sinh\theta = \cosh\theta \; \tanh\theta = \pm\beta\gamma$$

(12) Thus, the Lorentz transformation is

$$z' = \gamma z \pm \beta\gamma \; ct$$
$$ct' = \pm\beta\gamma \; z + \gamma \; ct$$
$$x' = x$$
$$y' = y$$

(13) As a matrix transformation we have

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

(14) We note that

$$L = \begin{pmatrix} \gamma & \pm\beta\gamma \\ \pm\beta\gamma & \gamma \end{pmatrix} \qquad L^{-1} = \begin{pmatrix} \gamma & \mp\beta\gamma \\ \mp\beta\gamma & \gamma \end{pmatrix}$$

(15) <u>Time dilation</u>: A clock is at rest in its own frame (thus does not move).

$$\begin{pmatrix} \Delta z' \\ c\Delta t' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \underbrace{\begin{pmatrix} \boxed{\Delta z \neq 0} \\ c\Delta t \end{pmatrix}}_{\text{rest frame of clock.}}$$

$$c\,\Delta t' = \gamma\, c\,\Delta t$$

$$T' = \gamma T$$

$$T' \geq T \quad \text{(time dilates!)}$$

(16) <u>Length contraction</u>: Length of a rod is measured in a moving frame such that $\Delta t = 0$ for the events measuring the two points on the rod.

$$\begin{pmatrix} \Delta z' \\ \boxed{c\Delta t' \neq 0} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \underbrace{\begin{pmatrix} \Delta z \\ c\Delta t \end{pmatrix}}_{\text{rest frame of rod.}}$$

$$\underbrace{\begin{pmatrix} \Delta z \\ c\Delta t \end{pmatrix}}_{\text{rest frame of rod.}} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} \Delta z' \\ c\Delta t' = 0 \end{pmatrix}$$

$$\Delta z = \gamma\, \Delta z'$$

$$\frac{L}{\gamma} = L'$$

$$L' < L$$

(length contracts!)