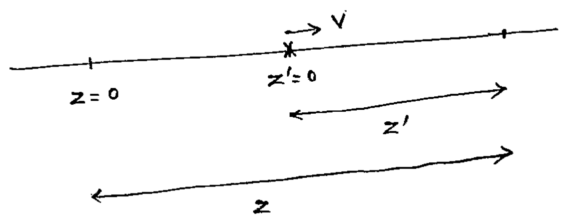
① Lorentz transformation (for boosts along z-direction) is

$$\begin{pmatrix} z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} z \\ ct \end{pmatrix}$$

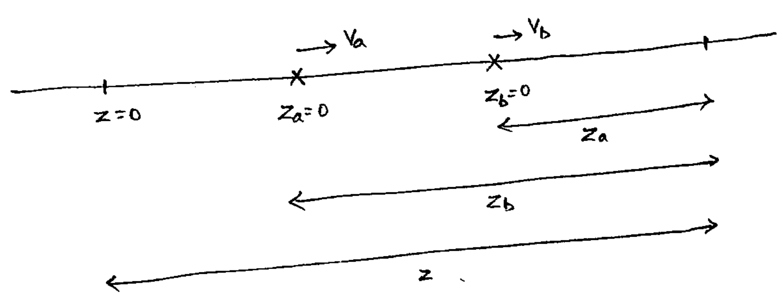
$x' = x$

$y' = y$



$z' = z - vt$

② Velocity addition



$$\begin{pmatrix} z_a \\ ct_a \end{pmatrix} = \begin{pmatrix} \gamma_a & -\beta_a\gamma_a \\ -\beta_a\gamma_a & \gamma_a \end{pmatrix} \begin{pmatrix} z \\ ct \end{pmatrix}$$

$$\begin{pmatrix} z_b \\ ct_b \end{pmatrix} = \begin{pmatrix} \gamma_b & -\beta_b\gamma_b \\ -\beta_b\gamma_b & \gamma_b \end{pmatrix} \begin{pmatrix} z \\ ct \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} z_a \\ ct_a \end{pmatrix} = \begin{pmatrix} \gamma_a & -\beta_a\gamma_a \\ -\beta_a\gamma_a & \gamma_a \end{pmatrix} \begin{pmatrix} \gamma_b & -\beta_b\gamma_b \\ -\beta_b\gamma_b & \gamma_b \end{pmatrix} \begin{pmatrix} z_b \\ ct_b \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_a\gamma_b(1+\beta_a\beta_b) & -(\beta_a+\beta_b)\gamma_a\gamma_b \\ -(\beta_a+\beta_b)\gamma_a\gamma_b & \gamma_a\gamma_b(1+\beta_a\beta_b) \end{pmatrix} \begin{pmatrix} z_b \\ ct_b \end{pmatrix}$$

(3) Comparing with

$$\begin{pmatrix} z_a \\ ct_a \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} z_b \\ ct_b \end{pmatrix}$$

we learn

$$\beta\gamma = (\beta_a + \beta_b)\,\gamma_a\gamma_b \qquad \text{— (i)}$$

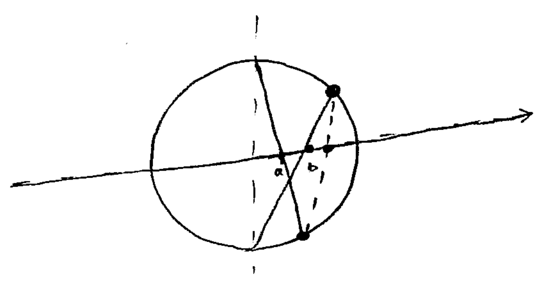
$$\gamma = (1 + \beta_a\beta_b)\,\gamma_a\gamma_b \qquad \text{— (ii)}$$

which when divided leads to the velocity addition formula

$\beta = \frac{v}{c}$, $\beta_a = \frac{v_a}{c}$, $\beta_b = \frac{v_b}{c}$.

$$\beta = \frac{\beta_a + \beta_b}{1 + \beta_a\beta_a}$$

(4) Jerzy Kocik's geometric diagram for velocity addition:



→ Presuming velocity addition holds for subluminal and superluminal this diagram helps in analysis.

5 Lorentz transformation leaves the following "distance" invariant

$$-\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$
$$= -c^2 \Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2.$$

For infinitely small displacements we have

$$-ds^2 = c^2 dt^2 + dx^2 + dy^2 + dz^2$$

6 We make the observation that

$$-ds^2 = \begin{pmatrix} -cdt & dx & dy & dz \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix}$$

7 We define the components of a contravariant and covariant vector as

$$x^\mu = (x^0, x^1, x^2, x^3) \equiv (ct, \vec{x}) \rightarrow \text{contravariant vector}$$
$$x_\mu = (x_0, x_1, x_2, x_3) \equiv (-ct, \vec{x}) \rightarrow \text{covariant vector}$$

8 Greek indices: μ, ν, α, β : 0, 1, 2, 3

Latin indices : i, j, m, n : 1, 2, 3

(9) We introduce the metric tensor, whose components are,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \delta^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

such that

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}{}_{\nu} .$$

(10) The metric tensor transform a contravariant vector into a covariant vector or vice versa.

$$x_\mu = g_{\mu\nu} x^\nu$$
$$x^\mu = g^{\mu\nu} x_\nu$$

(11) Thus,

$$x_0 = g_{0\nu} x^\nu = g_{00} x^0 + g_{01} x^1 + g_{02} x^2 + g_{03} x^3$$
$$= g_{00} x^0 = - x^0$$

$$x_1 = g_{1\nu} x^\nu = g_{11} x^1 = x^1$$
$$x_2 = g_{2\nu} x^\nu = g_{22} x^2 = x^2$$
$$x_3 = g_{3\nu} x^\nu = g_{33} x^3 = x^3$$