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(1) $x^\mu = (ct, \vec{x}) \longrightarrow dx^\mu dx_\mu = -c^2 dt^2 + d\vec{x}\cdot d\vec{x} = -ds^2$

$$u^\alpha = c\,\frac{dx^\alpha}{ds}$$

$$= (c\gamma, \vec{v}\gamma) \longrightarrow u^\alpha u_\alpha = -c^2$$

Physics

$$\left(\frac{E}{c}, \vec{P}\right) \leftarrow p^\alpha = m u^\alpha = (mc\gamma, m\vec{v}\gamma)$$

$$c\frac{dt}{ds} = \gamma$$

(2)
$$a^\alpha = c\,\frac{du^\alpha}{ds}$$

$$= \frac{c\,dt}{ds}\frac{d}{dt}u^\alpha$$

$$= \gamma\frac{d}{dt}(c\gamma, \vec{v}\gamma)$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \gamma\left(c\frac{d\gamma}{dt}, \frac{d\vec{v}}{dt}\gamma + \vec{v}\frac{d\gamma}{dt}\right)$$

$$= \gamma\left(c\frac{d\gamma}{dt}, \vec{a}\gamma + \vec{v}\frac{d\gamma}{dt}\right)$$

(3) Observe that starting from

$$u^\alpha u_\alpha = -c^2$$

we can show that (by differentiating with respect to s)

$$a^\alpha u_\alpha = 0.$$

HW: Is a^α a space-like or time-like quantity?

(4) Thus, using

$$u^\alpha = \gamma (c, \vec{v})$$

$$a^\alpha = \gamma \left(c \frac{d\gamma}{dt}, \; \vec{a} \gamma + \vec{v} \frac{d\gamma}{dt} \right)$$

we have

$$a^\alpha u_\alpha = 0$$

$$\gamma^2 \left[- c^2 \frac{d\gamma}{dt} + \vec{v}.\vec{a} \gamma + \vec{v}.\vec{v} \frac{d\gamma}{dt} \right] = 0$$

$$- c^2 \gamma^2 \left[\left(1 - \frac{v^2}{c^2}\right) \frac{d\gamma}{dt} - \frac{\vec{v}.\vec{a}}{c^2} \gamma \right] = 0$$

$$1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2}$$

$$- c^2 \left[\frac{d\gamma}{dt} - \frac{\vec{v}.\vec{a}}{c^2} \gamma^3 \right] = 0$$

$$\Rightarrow \frac{d\gamma}{dt} = \frac{\vec{v}.\vec{a}}{c^2} \gamma^3$$

(5) Using (4) in the expression of a^α in (4) we have.

$$a^\alpha = \gamma \left(c \frac{d\gamma}{dt}, \; \vec{a} \gamma + \vec{v} \frac{d\gamma}{dt} \right)$$

$$= \gamma \left(\frac{\vec{v}.\vec{a}}{c} \gamma^3, \; \vec{a} \gamma + \vec{v} \frac{\vec{v}.\vec{a}}{c^2} \gamma^3 \right)$$

(6) Let us next define 4-force as

$$f^\alpha = c \frac{d}{ds} P^\alpha$$

$$= \frac{c\,dt}{ds} \frac{d}{dt} P^\alpha$$

$$= \gamma \frac{d}{dt} \left(\frac{E}{c}, \vec{P} \right)$$

$$= \gamma \left(\frac{1}{c} \frac{dE}{dt}, \frac{d\vec{P}}{dt} \right)$$

(7) Relativistic equation of motion is given using

$$f^\alpha = m\, a^\alpha$$

$$\gamma \left(\frac{1}{c} \frac{dE}{dt}, \frac{d\vec{P}}{dt} \right) = m \gamma \left(c \frac{d\gamma}{dt}, \vec{a}\,\gamma + \vec{v} \frac{d\gamma}{dt} \right)$$

$$= m \gamma \left(\frac{\vec{v}\cdot\vec{a}}{c} \gamma^3, \vec{a}\,\gamma + \vec{v} \frac{\vec{v}\cdot\vec{a}}{c^2} \gamma^3 \right).$$

This implies

$$\frac{dE}{dt} = \frac{d}{dt}(mc\gamma) = m \frac{\vec{v}\cdot\vec{a}}{c} \gamma^3$$

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}(m\vec{v}\gamma) = m\vec{a}\,\gamma + m\vec{v} \frac{\vec{v}\cdot\vec{a}}{c^2} \gamma^3$$

8 Let us calculate $a^\alpha a_\alpha$, which will remain invariant between observers.

$$a^\alpha = \gamma\left(\frac{\vec{v}\cdot\vec{a}}{c}\gamma^3,\ \vec{a}\,\gamma + \vec{v}\,\frac{\vec{v}\cdot\vec{a}}{c^2}\gamma^3\right)$$

$$a^\alpha a_\alpha = \gamma^2\left[-\left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2\gamma^6 + \left(\vec{a}\,\gamma + \vec{v}\,\frac{\vec{v}\cdot\vec{a}}{c^2}\gamma^3\right)^2\right]$$

$$= \gamma^2\left[-\left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2\gamma^6 + \vec{a}\cdot\vec{a}\,\gamma^2 + \frac{v^2}{c^2}\left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2\gamma^6 + 2\left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2\gamma^4\right]$$

$$= \gamma^2\left[\vec{a}\cdot\vec{a}\,\gamma^2 + \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2\gamma^4\left\{2 + \underbrace{\beta^2\gamma^2 - \gamma^2}_{-1}\right\}\right]$$

$$= \vec{a}\cdot\vec{a}\,\gamma^4 + \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2\gamma^6$$

9 Using $(\vec{v}\times\vec{a})\cdot(\vec{v}\times\vec{a}) = \vec{v}\cdot\vec{v}\ \vec{a}\cdot\vec{a} - (\vec{v}\cdot\vec{a})^2$

we can also write

$$a^\alpha a_\alpha = \vec{a}\cdot\vec{a}\,\gamma^4 + \left[\frac{v^2}{c^2}\,\vec{a}\cdot\vec{a} - \left(\frac{\vec{v}\times\vec{a}}{c}\right)\cdot\left(\frac{\vec{v}\times\vec{a}}{c}\right)\right]\gamma^6$$

$$= \vec{a}\cdot\vec{a}\,\gamma^4\underbrace{\left(1-\beta^2\gamma^2\right)}_{\gamma^2} - \left(\frac{\vec{v}\times\vec{a}}{c}\right)\cdot\left(\frac{\vec{v}\times\vec{a}}{c}\right)\gamma^6$$

$$= \vec{a}\cdot\vec{a}\,\gamma^6 - \left(\frac{\vec{v}\times\vec{a}}{c}\right)\cdot\left(\frac{\vec{v}\times\vec{a}}{c}\right)\gamma^6$$

(10) Using (8) and (9) we conclude.

$$\vec{v} \perp \vec{a} : \qquad a^\alpha a_\alpha = \vec{a}\cdot\vec{a}\ r^4$$

$$\vec{v} \parallel \vec{a} : \qquad a^\alpha a_\alpha = \vec{a}\cdot\vec{a}\ r^6$$

Motion in a magnetic field (under simple initial conditions) corresponds to the case $\vec{v} \perp \vec{a}$, and motion in an electric field (under simple initial conditions) corresponds to the case $\vec{v} \parallel \vec{a}$.