Hyperbolic motion (constant proper acceleration)

1 Summary

$$x^\alpha = (ct, \vec{x}) \longrightarrow dx^\alpha dx_\alpha = -c^2 dt^2 + d\vec{x}\cdot d\vec{x} = -ds^2$$

$$u^\alpha = c\frac{d}{ds}x^\alpha$$
$$= (c\gamma, \vec{v}\gamma) \longrightarrow u^\alpha u_\alpha = -c^2$$

$$p^\alpha = m u^\alpha \longrightarrow p^\alpha p_\alpha = -m^2c^2$$

$$a^\alpha = c\frac{d}{ds}u^\alpha$$
$$= \gamma\left(\frac{d}{dt}(c\gamma), \frac{d}{dt}(\vec{v}\gamma)\right) \rightarrow a^\alpha a_\alpha = \vec{a}\cdot\vec{a}\,\gamma^4 + \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2\gamma^6$$
$$= \vec{a}\cdot\vec{a}\,\gamma^6 - \left(\frac{\vec{v}\times\vec{a}}{c}\right)\cdot\left(\frac{\vec{v}\times\vec{a}}{c}\right)\gamma^6$$

$$f^\alpha = m a^\alpha \longrightarrow f^\alpha f_\alpha = m^2 a^\alpha a_\alpha$$

Physics:

$$\left(\frac{E}{c}, \vec{P}\right) \equiv p^\alpha = (mc\gamma, m\vec{v}\gamma)$$

which leads to the equations of motion

$$\frac{dE}{dt} = \frac{d}{dt}(mc^2\gamma) = m\vec{a}\cdot\vec{v}\,\gamma^3$$

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(m\vec{v}\gamma) = m\vec{a}\,\gamma + m\vec{v}\,\frac{\vec{v}\cdot\vec{a}}{c^2}\,\gamma^3$$

Also, note that

$$\frac{dE}{dt} = \vec{F}\cdot\vec{v}$$

which is valid in Newtonian mechanics too.

2 Let us consider the motion of a particle undergoing constant proper acceleration

$$a^\alpha a_\alpha = a^2.$$

$a \to$ invariant under Lorentz transformation.

Let us further assume the motion to be in a straight line. Thus

$$a^\alpha a_\alpha = \vec{a}\cdot\vec{a}\ \gamma^6 - \underbrace{\cancel{\left(\frac{\vec{v}\times\vec{a}}{c}\right)}}_{=0}\cdot\left(\frac{\vec{v}\times\vec{a}}{c}\right)\gamma^6$$

$$|a^\alpha|^2 = \vec{a}\cdot\vec{a}\ \gamma^6$$

$$|a^\alpha| = |\vec{a}|\ \gamma^3$$

3 We begin with

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}\gamma) = m\vec{a}\gamma + m\vec{v}\,\frac{\vec{v}\cdot\vec{a}}{c^2}\gamma^3.$$

Supressing the vector notation because everything is happening in the direction of motion, we have

$$\frac{d}{dt}(v\gamma) = |\vec{a}|\,\gamma\underbrace{\left(1+\beta^2\gamma^2\right)}_{\gamma^2}$$

$$= |\vec{a}|\,\gamma^3$$

$$= |a^\alpha|$$

④ Integrating

$$\frac{d}{dt}(v\,r) = |a^\alpha|$$

we obtain

$$v\,r - (v\,r)_{t=0} = |a^\alpha|\,t$$

⑤ Let $v = 0$ at $t = 0$. Thus,

$$v\,r = |a^\alpha|\,t$$

$$\frac{v^2}{1-\frac{v^2}{c^2}} = |a^\alpha|^2 t^2$$

$$\frac{1}{c}\frac{dx}{dt} = \frac{v}{c} = \frac{\frac{|a^\alpha|\,t}{c}}{\sqrt{1+\frac{|a^\alpha|^2}{c^2}t^2}}$$

⑥ Thus,

$$x(t) - x(0) = \int_0^t \frac{|a^\alpha|\,t'\,dt'}{\sqrt{1+\frac{|a^\alpha|^2}{c^2}t'^2}}$$

$1-u^2 = v$

$$= \frac{c^2}{|a^\alpha|}\int_0^{\frac{|a^\alpha|}{c}t} \frac{u\,du}{\sqrt{1+u^2}}$$

$$= \frac{c^2}{|a^\alpha|}\left[\sqrt{1+\frac{|a^\alpha|^2}{c^2}t^2} - 1\right]$$

⑦ Further, let $x(0) = \frac{c^2}{|a^\alpha|}$, then

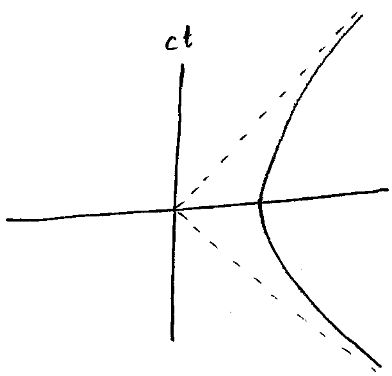
$$x(t) = \frac{c^2}{|a^\alpha|}\sqrt{1+\frac{|a^\alpha|^2}{c^2}t^2}$$

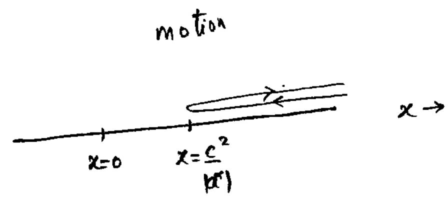
8 Thus,

$$x(t)^2 = \frac{c^4}{|a^\alpha|^2} + c^2 t^2 .$$

$$x(t)^2 - c^2 t^2 = \frac{c^4}{|a^\alpha|^2},$$

which is the equation of a hyperbola.





9 Non-relativistic limit

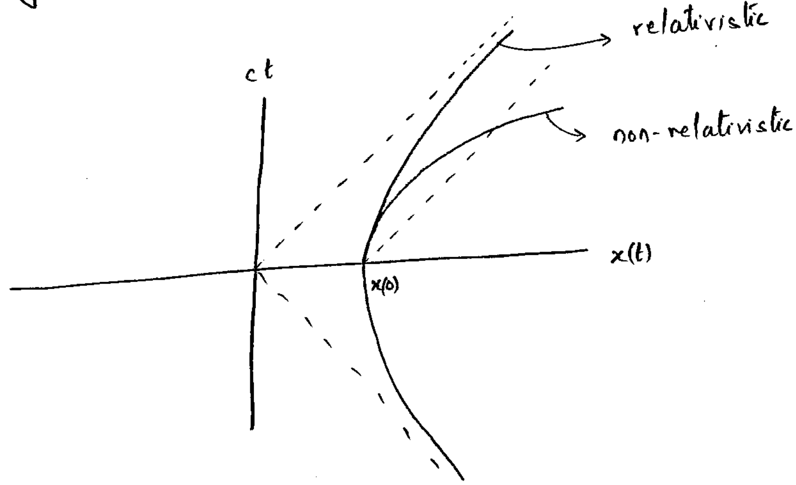
The relativistic position,

$$x(t) = \frac{c^2}{|a^\alpha|}\sqrt{1 + \frac{|a^\alpha|^2}{c^2} t^2}$$

$$\xrightarrow{t \ll \frac{c}{|a^\alpha|}} \underbrace{\frac{c^2}{|a^\alpha|}}_{x(0)} + \frac{1}{2} |a^\alpha| t^2$$

which is the position, for small times, in the relativistic limit.

(10) Plotting 'ct' as a function of '$x(t)$'



$$x(0) = \frac{c^2}{|a^\alpha|}$$

Relativistic : $ct = \sqrt{x(t)^2 - x(0)^2} \xrightarrow{x(0) \ll x(t)} x(t)$

Non-relativistic : $t = \sqrt{\frac{2}{|a^\alpha|}\left(x(t) - x(0)\right)} \xrightarrow{x(0) \ll x(t)} \sqrt{x(t)}$

Note that when the slope $c\frac{dt}{dx} < 1$ corresponds to superluminal speeds.