Midterm Exam No. 02 (Spring 2015)

PHYS 520B: Electromagnetic Theory

Date: 2014 Mar 18

1. (**20 points.**) The vector potential for a point magnetic moment $\mathbf{m}$ is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}. \tag{1}$$

Verify that the magnetic field due to the point dipole given by

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{2}$$

satisfies

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{3}$$

2. (**30 points.**) Consider a straight wire of radius a carrying current I described using the current density

$$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{z}} \, \frac{C}{\rho} \, e^{-\lambda\rho} \, \theta(a - \rho), \tag{4}$$

where $\theta(x) = 1$ for $x > 1$ and zero otherwise.

   (a) Find C in terms of the current I.

   (b) Find the magnetic field inside and outside the wire.

   (c) Plot the magnetic field as a function of ρ.

3. (**30 points.**) The current density for a wire forming a helix and carrying a steady current I is given by

$$\mathbf{J}(\mathbf{r}) = \mathbf{n} \, I \sum_{m=-\infty}^{\infty} \frac{1}{\rho} \delta(\rho - a) \delta\left(\phi - 2\pi \frac{z}{L} + 2\pi(m - 1)\right), \tag{5}$$

where the direction of the flow of current is described by the vector

$$\mathbf{n} = \hat{\mathbf{z}} + 2\pi \frac{a}{L} \, \hat{\boldsymbol{\phi}}. \tag{6}$$

Here a is the radius of the helix and L is the pitch. The coordinates (ρ, ϕ, z) are the usual cylindrical coordinates. The coordinate ϕ generates one period of the helix for $0 < \phi < 2\pi$, and the sum periodically repeats it.

(a) Calculate the flux of current density

$$\int_S d\mathbf{a} \cdot \mathbf{J}(\mathbf{r}) \tag{7}$$

passing through the surface S of the $z = 0$ plane.
Hint: The area element for the $z = 0$ plane is $d\mathbf{a} = \hat{\mathbf{z}}\, dx\, dy = \hat{\mathbf{z}}\, \rho d\rho\, d\phi$.

(b) Using

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \tag{8}$$

it is possible to determine the magnetic field on the symmetry axis of the helix in terms of the modified Bessel functions,

$$\mathbf{B}(0, 0, z) = \frac{\mu_0 I}{L} \left[\hat{\mathbf{z}} - \hat{\boldsymbol{\phi}} \left\{ \frac{2\pi a}{L} K_0 \left(\frac{2\pi a}{L} \right) + K_1 \left(\frac{2\pi a}{L} \right) \right\} \right]. \tag{9}$$

How is the $\hat{\mathbf{z}}$-component of the magnetic field related to the magnetic field of a solenoid?

(c) Using the fact that the modified Bessel functions for large arguments tends to zero determine the magnetic field on the z-axis in this limit. How well does a helix with pitch L small compared to radius a compare with a solenoid?