Final Exam (Spring 2015)

PHYS 520B: Electromagnetic Theory

Date: 2015 May 15

1. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration g is described by the equation of a hyperbola

$$z_q(t) = \sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{1}$$

This is the motion of a particle that comes to existance at $z_q = +\infty$ at $t = -\infty$, then 'falls' in a region of constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = z_0$, and then returns back to infinity. Assume you are positioned at the origin. If the particle is a source of light (imagine a flash light) at what time will the light first reach you at the origin?

2. (**40 points.**) The electric and magnetic fields transform under a Lorentz transformation (for boost in z direction) as

$$E'_x(\mathbf{r}', t') = \gamma\, E_x(\mathbf{r}, t) + \beta\gamma\, cB_y(\mathbf{r}, t), \tag{2a}$$
$$cB'_y(\mathbf{r}', t') = \beta\gamma\, E_x(\mathbf{r}, t) + \gamma\, cB_y(\mathbf{r}, t), \tag{2b}$$
$$E'_z(\mathbf{r}', t') = E_z(\mathbf{r}, t) \tag{2c}$$
$$cB'_x(\mathbf{r}', t') = \gamma\, cB_x(\mathbf{r}, t) - \beta\gamma\, E_y(\mathbf{r}, t), \tag{3a}$$
$$E'_y(\mathbf{r}', t') = -\beta\gamma\, cB_x(\mathbf{r}, t) + \gamma\, E_y(\mathbf{r}, t), \tag{3b}$$
$$cB'_z(\mathbf{r}', t') = cB_z(\mathbf{r}, t), \tag{3c}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. The transformed values of the coordinates and the fields are distinguished by a prime. Derive the invariance properties

$$\mathbf{E}'(\mathbf{r}', t') \cdot \mathbf{B}'(\mathbf{r}', t') = \mathbf{E}(\mathbf{r}, t) \cdot \mathbf{B}(\mathbf{r}, t) \tag{4}$$

and

$$\mathbf{E}'(\mathbf{r}', t')^2 - c^2\mathbf{B}'(\mathbf{r}', t')^2 = \mathbf{E}(\mathbf{r}, t)^2 - c^2\mathbf{B}(\mathbf{r}, t)^2. \tag{5}$$

3. (**40 points.**) Consider a particle of charge q moving along the path $\mathbf{r}_q(t)$. The corresponding charge density and current density are

$$\rho(\mathbf{r}', t') = q\, \delta^{(3)}(\mathbf{r} - \mathbf{r}_q(t')), \tag{6a}$$
$$\mathbf{j}(\mathbf{r}', t') = q\, \mathbf{v}_q(t')\delta^{(3)}(\mathbf{r} - \mathbf{r}_q(t')), \tag{6b}$$

where $\mathbf{v}_q(t)$ is the velocity of the particle at time t.

(a) Beginning from

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \int_{-\infty}^{\infty} dt' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t - t' - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right), \tag{7a}$$
$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' \int_{-\infty}^{\infty} dt' \frac{\mathbf{j}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t - t' - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right), \tag{7b}$$

and using Eqs. (6) derive

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} dt' \, \frac{\delta\left(t-t'-\frac{1}{c}|\mathbf{r}-\mathbf{r}_q(t')|\right)}{|\mathbf{r}-\mathbf{r}'|}, \tag{8a}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} dt' \, q\mathbf{v}_q(t') \, \frac{\delta\left(t-t'-\frac{1}{c}|\mathbf{r}-\mathbf{r}_q(t')|\right)}{|\mathbf{r}-\mathbf{r}'|}. \tag{8b}$$

(b) Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{9}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$, evaluate the integrals (requiring the roots to be causal, that is, $t_r < t$) in Eqs. (8) as

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\left[|\mathbf{r}-\mathbf{r}(t_r)| - \frac{\mathbf{v}_q(t_r)}{c}\cdot\Big\{\mathbf{r}-\mathbf{r}_q(t_r)\Big\}\right]}, \tag{10a}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{q\mathbf{v}_q(t')}{\left[|\mathbf{r}-\mathbf{r}(t_r)| - \frac{\mathbf{v}_q(t_r)}{c}\cdot\Big\{\mathbf{r}-\mathbf{r}_q(t_r)\Big\}\right]}, \tag{10b}$$

where t_r is uniquely determined using

$$F(t_r) = c(t-t_r) - |\mathbf{r}-\mathbf{r}(t_r)| = 0, \qquad t_r < t. \tag{11}$$

(c) In terms of the four-vectors

$$x^\alpha - x_q^\alpha(t_r) = (ct - ct_r, \mathbf{r}-\mathbf{r}_q(t_r)) \tag{12}$$

and

$$u_q^\alpha = \gamma_q(c, \mathbf{v}_q(t_r)), \qquad \gamma_q = \frac{1}{\sqrt{1-\frac{\mathbf{v}_q(t_r)^2}{c^2}}}, \tag{13}$$

show that the expression in the denominator can be interpreted as

$$-\frac{1}{c\gamma_q}(u_q)_\alpha(x^\alpha - x_q^\alpha(t_r)) = c(t-t_r) - \frac{\mathbf{v}_q(t_r)}{c}\cdot\Big\{\mathbf{r}-\mathbf{r}_q(t_r)\Big\} \tag{14a}$$

$$= |\mathbf{r}-\mathbf{r}(t_r)| - \frac{\mathbf{v}_q(t_r)}{c}\cdot\Big\{\mathbf{r}-\mathbf{r}_q(t_r)\Big\}. \tag{14b}$$

Thus, $F(t_r) = 0$ implies

$$(u_q)_\alpha(x^\alpha - x_q^\alpha(t_r)) = 0, \tag{15}$$

stating that these events are separated by light-like distance.