

Homework No. 03 (Spring 2015)

PHYS 520B: Electromagnetic Theory

Due date: Monday, 2015 Feb 20, 4.30pm

1. **(20 points.)** The solution to the Maxwell equations for the case of magnetostatics was found in terms of the vector potential \mathbf{A} to be

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (1)$$

- (a) Verify that the above solution satisfies the Coulomb gauge condition, that is, it satisfies

$$\nabla \cdot \mathbf{A} = 0. \quad (2)$$

- (b) Further, verify that the magnetic field is the curl of the vector potential and can be expressed in the form

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (3)$$

2. **(20 points.)** The solution to the Maxwell equations for the case of magnetostatics was found to be

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (4)$$

Verify that the above solution satisfies magnetostatics equations, that is, it satisfies

$$\nabla \cdot \mathbf{B} = 0 \quad (5)$$

and

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \quad (6)$$

3. **(50 points.)** (Based on Problem 5.8, Griffiths 4th edition.)

The magnetic field at position $\mathbf{r} = (x, y, z)$ due to a finite wire segment of length $2L$ carrying a steady current I , with the caveat that it is unrealistic (why?), placed on the z -axis with its end points at $(0, 0, L)$ and $(0, 0, -L)$, is

$$\mathbf{B}(\mathbf{r}) = \hat{\phi} \frac{\mu_0 I}{4\pi} \frac{1}{\sqrt{x^2 + y^2}} \left[\frac{z + L}{\sqrt{x^2 + y^2 + (z + L)^2}} - \frac{z - L}{\sqrt{x^2 + y^2 + (z - L)^2}} \right], \quad (7)$$

where $\hat{\phi} = (-\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}}) = (-y \hat{\mathbf{i}} + x \hat{\mathbf{j}}) / \sqrt{x^2 + y^2}$.

- (a) Show that by taking the limit $L \rightarrow \infty$ we obtain the magnetic field near a long straight wire carrying a steady current I ,

$$\mathbf{B}(\mathbf{r}) = \hat{\phi} \frac{\mu_0 I}{2\pi\rho}, \quad (8)$$

where $\rho = \sqrt{x^2 + y^2}$ is the perpendicular distance from the wire.

- (b) Show that the magnetic field on a line bisecting the wire segment is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\phi} \frac{\mu_0 I}{2\pi\rho} \frac{L}{\sqrt{\rho^2 + L^2}}. \quad (9)$$

- (c) Find the magnetic field at the center of a square loop, which carries a steady current I . Let $2L$ be the length of a side, ρ be the distance from center to side, and $R = \sqrt{\rho^2 + L^2}$ be the distance from center to a corner. (Caution: Notation differs from Griffiths.) You should obtain

$$B = \frac{\mu_0 I}{2R} \frac{4}{\pi} \tan \frac{\pi}{4}. \quad (10)$$

- (d) Show that the magnetic field at the center of a regular n -sided polygon, carrying a steady current I is

$$B = \frac{\mu_0 I}{2R} \frac{n}{\pi} \tan \frac{\pi}{n}, \quad (11)$$

where R is the distance from center to a corner of the polygon.

- (e) Show that the magnetic field at the center of a circular loop of radius R ,

$$B = \frac{\mu_0 I}{2R}, \quad (12)$$

is obtained in the limit $n \rightarrow \infty$.

4. **(40 points.)** (Refer Schwinger et al. problem 26.1 and the article in Ref. [1].)
A simple model of a metal describes the electrons in it using Newton's law,

$$m \frac{d^2 \mathbf{x}}{dt^2} + m\gamma \frac{d\mathbf{x}}{dt} + m\omega_0^2 \mathbf{x} = e\mathbf{E}. \quad (13)$$

Here the first term involves the acceleration of electron, ω_0 -term binds the electron to the atoms, while γ -term damps the motion. Conductivity in typical metals is dominated by the damping term, thus

$$m\gamma \mathbf{v} = e\mathbf{E}. \quad (14)$$

The current density \mathbf{j} for (constant) density n_f of conduction electrons is

$$\mathbf{j} = n_f e \mathbf{v}. \quad (15)$$

In conjunction we have

$$\mathbf{j} = \frac{n_f e^2}{m\gamma} \mathbf{E} = \sigma \mathbf{E}, \quad (16)$$

where σ is the static conductivity.

In 1935 Fritz London and Heinz London proposed that the current density \mathbf{j}_s in a superconductor is described by the acceleration term

$$m \frac{d\mathbf{v}}{dt} = e\mathbf{E}, \quad (17)$$

which leads to London's "acceleration equation"

$$\mu_0 \frac{d\mathbf{j}_s}{dt} = \mu_0 \frac{n_f e^2}{m} \mathbf{E} = \frac{1}{\lambda_L^2} \mathbf{E}. \quad (18)$$

In terms of potentials we obtain the London equation

$$\mu_0 \mathbf{j}_s + \frac{1}{\lambda_L^2} \mathbf{A} = \nabla \chi, \quad (19)$$

where \mathbf{A} is the vector potential and χ allows for a choice of gauge.

- (a) Using London's equation show that a superconductor is characterized by the equations

$$\mu_0 \frac{\partial \mathbf{j}_s}{\partial t} = \frac{1}{\lambda_L^2} \mathbf{E}, \quad (20)$$

$$\mu_0 \nabla \times \mathbf{j}_s = -\frac{1}{\lambda_L^2} \mathbf{B}. \quad (21)$$

(Hint: Choose the scalar potential $\phi = -\lambda_L^2 \partial \chi / \partial t$.)

- (b) Show that the magnetic field satisfies the equation

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{B} = \frac{1}{\lambda_L^2} \mathbf{B}. \quad (22)$$

- (c) For the static case, $\partial \mathbf{B} / \partial t = 0$, show that

$$\nabla^2 \mathbf{B} = \frac{1}{\lambda_L^2} \mathbf{B}, \quad (23)$$

which implies the Meissner effect, that a uniform magnetic field cannot exist inside a superconductor.

- (d) In this static limit and planar geometry, it implies

$$\mathbf{B} = \mathbf{B}_0 e^{-\frac{x}{\lambda_L}}, \quad (24)$$

where λ_L is called the London penetration depth,

$$\lambda_L^2 = \frac{m\epsilon_0}{n_f e^2} c^2. \quad (25)$$

Calculate the penetration depth for $n_f \sim 6 \times 10^{28} / \text{m}^3$ (electron number density for gold) and show that it is of the order of tens of nanometers.

References

- [1] F. London and H. London. The electromagnetic equations of the superconductor. *Proc. R. Soc. London A: Mathematical, Physical and Engineering Sciences*, 149(866):71–88, 1935.