Homework No. 04 (Spring 2015)

PHYS 520B: Electromagnetic Theory

Due date: Monday, 2015 Mar 2, 4.30pm

1. (**50 points.**) Consider a wire segment of length $2L$ carrying a steady current I, described by

$$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{z}} I \delta(x)\delta(y)\theta(-L < z < L), \tag{1}$$

when the rod is placed on the z-axis centered on the origin. Here $\theta(-L < z < L) = 0$, if $z > L$ and $z < -L$, and $\theta(-L < z < L) = 1$, otherwise.

(a) Show that the vector potential of the wire is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\, I \left[\sinh^{-1}\left(\frac{L-z}{\sqrt{x^2+y^2}} \right) + \sinh^{-1}\left(\frac{L+z}{\sqrt{x^2+y^2}} \right) \right]. \tag{2}$$

(b) Show that

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}). \tag{3}$$

Also, using Eq. (3), verify that

$$\sinh^{-1}(-x) = -\sinh^{-1} x. \tag{4}$$

(c) Thus, express the vector potential of Eq. (2) in the form

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\, I \left[-2\ln\frac{R}{L} + F\left(\frac{z}{L}, \frac{R}{L} \right) \right], \tag{5}$$

where $R^2 = x^2 + y^2$ and

$$F(a,b) = \ln[1 - a + \sqrt{(1-a)^2 + b^2}] + \ln[1 + a + \sqrt{(1+a)^2 + b^2}]. \tag{6}$$

(d) Show that

$$\mathbf{A}(\mathbf{r}) \xrightarrow{R \ll L, z \ll L} -\frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\, 2I \ln\frac{R}{2L}. \tag{7}$$

(e) Using $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ determine the magnetic field for an infinite rod (placed on the z-axis) to be

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{2I}{R}\hat{\boldsymbol{\phi}}. \tag{8}$$

2. (**40 points.**) A circular loop of wire carries a charge q. It rotates with angular velocity $\boldsymbol{\omega}$ about its axis, say z-axis.

(a) Show that the current density generated by this motion is given by

$$\mathbf{J}(\mathbf{r}) = \frac{q}{2\pi a}\,\boldsymbol{\omega}\times\mathbf{r}\,\delta(\rho - a)\delta(z-0). \tag{9}$$

Hint: Use $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$, and $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ for circular motion.

(b) Using

$$\mathbf{m} = \frac{1}{2}\int d^3r\,\mathbf{r}\times\mathbf{J}(\mathbf{r}). \tag{10}$$

determine the magnetic dipole moment of this loop to be

$$\mathbf{m} = \frac{qa^2}{2}\boldsymbol{\omega}. \tag{11}$$

(c) Calculate the vector potential $\mathbf{A}(0,0,z)$ on the z-axis.

(d) Calculate the magnetic field $\mathbf{B}(0,0,z)$ on the z-axis.