Homework No. 05 (Spring 2015)

PHYS 520B: Electromagnetic Theory

Due date: None

1. (**20 points.**) Verify the following:

$$\begin{aligned}
\mathrm{Tr}A &= A_i{}^i. && \text{(1a)}\\
\det A &= \varepsilon_{i_1 i_2 \ldots i_n} A^{i_1}{}_1 A^{i_2}{}_2 \ldots A^{i_n}{}_n && \text{(1b)}\\
&= \frac{1}{n!}\varepsilon_{i_1 i_2 \ldots i_n}\varepsilon^{i'_1 i'_2 \ldots i'_n} A^{i_1}{}_{i'_1} A^{i_2}{}_{i'_2} \ldots A^{i_n}{}_{i'_n}, && \text{(1c)}
\end{aligned}$$

where n is the dimension of the matrix A.

2. (**20 points.**) Prove that any orthogonal matrix R satisfying

$$RR^T = 1 \tag{2}$$

in N-dimensions has $N(N-1)/2$ independent variables.

3. (**20 points.**) Lorentz transformation describing a boost in the x-direction, y-direction, and z-direction, are

$$L_1 = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} \gamma_2 & 0 & -\beta_2\gamma_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_2\gamma_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} \gamma_3 & 0 & 0 & -\beta_3\gamma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_3\gamma_3 & 0 & 0 & \gamma_3 \end{pmatrix}, \tag{3}$$

respectively. Transformation describing a rotation about the x-axis, y-axis, and z-axis, are

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\omega_1 & \sin\omega_1 \\ 0 & 0 & -\sin\omega_1 & \cos\omega_1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_2 & 0 & -\sin\omega_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\omega_2 & 0 & \cos\omega_2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_3 & \sin\omega_3 & 0 \\ 0 & -\sin\omega_3 & \cos\omega_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

respectively. For infinitesimal transformations use the approximations

$$\gamma_i \sim 1, \qquad \cos\omega_i \sim 1, \qquad \sin\omega_i \sim \omega_i, \tag{5}$$

to derive

$$\Big[L_1, L_2\Big] = L_1 L_2 - L_2 L_1 = R_3 \quad \text{with} \quad \omega_3 = \beta_1\beta_2. \tag{6}$$

This states that boosts in perpendicular direction leads to rotation.

4. (**20 points.**) (Refer Hughston and Tod's book.)
Prove that

   (a) if p_μ is a time-like vector and $p^\mu s_\mu = 0$ then s^μ is necessarily space-like.

   (b) if p_μ and q^μ are both time-like vectors and $p^\mu q_\mu > 0$ then either both are future-pointing or both are past-pointing.

5. (**20 points.**) Non-relativistic limits are obtained for $\beta \ll 1$ in relativistic formulae.

   (a) Does Lorentz transformation lead to Galilean transformation for $\beta \ll 1$?

   (b) Does Lorentz transformation lead to Galilean transformation for $\beta \ll 1$ and $c \to \infty$ $(\Delta z \ll c\Delta t)$?

6. (**20 points.**) The path of a relativistic particle moving along a straight line with constant acceleration α is described by the equation of a hyperbola

$$x^2 - c^2t^2 = \frac{c^4}{\alpha^2}. \tag{7}$$

Will a photon dispatched to 'chase' this particle, at $t = 0$ from $x = 0$, ever catch up with it?

7. (**80 points.**) Eigenvalues of the field tennsor. (We choose $c = 1$, which is easily undone by replacing $\mathbf{E} \to \frac{1}{c}\mathbf{E}$ everywhere.)

   (a) Using

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{8}$$

   evaluate the following:

   i. $F^{\mu\lambda}F_{\lambda\nu}$

   ii. $\tilde{F}^{\mu\lambda}\tilde{F}_{\lambda\nu}$

   iii. Then, derive

$$F^{\mu\lambda}\tilde{F}_{\lambda\nu} = \delta^\mu{}_\nu \mathbf{E}\cdot\mathbf{B}, \tag{9a}$$

$$\tilde{F}^{\mu\lambda}\tilde{F}_{\lambda\nu} - F^{\mu\lambda}F_{\lambda\nu} = \delta^\mu{}_\nu (B^2 - E^2). \tag{9b}$$

   (b) Define

$$2f = (B^2 - E^2) \quad \text{and} \quad g = \mathbf{E}\cdot\mathbf{B}. \tag{10}$$

   Thus, construct matrix (or dyadic) equations

$$\mathbf{F}\cdot\tilde{\mathbf{F}} = g\mathbf{1}, \tag{11a}$$

$$\tilde{\mathbf{F}}\cdot\tilde{\mathbf{F}} - \mathbf{F}\cdot\mathbf{F} = 2f\mathbf{1}, \tag{11b}$$

   in terms of matrices (or dyadics) $\mathbf{F}$ and $\tilde{\mathbf{F}}$.

(c) Show that the eigenvalues λ of the field tensor $\mathbf{F}$ satisfy the quartic equation

$$\lambda^4 + 2f\lambda^2 - g^2 = 0. \tag{12}$$

(d) Evaluate the eigenvalues to be $\pm\lambda_1$ and $\pm\lambda_2$ where

$$\lambda_1 = \sqrt{-f - \sqrt{f^2 + g^2}} = \frac{i}{\sqrt{2}}\left[\sqrt{f + ig} + \sqrt{f - ig}\right], \tag{13}$$

$$\lambda_2 = \sqrt{-f + \sqrt{f^2 + g^2}} = \frac{i}{\sqrt{2}}\left[\sqrt{f + ig} - \sqrt{f - ig}\right]. \tag{14}$$

(e) Show that

i. if $B^2 - E^2 = 0$, then the eigenvalues are $\pm\sqrt{g}$ and $\pm i\sqrt{g}$.

ii. if $\mathbf{B} \cdot \mathbf{E} = 0$, then the eigenvalues are 0, 0, and $\pm\sqrt{2f}$.

(f) Prove the following:

i. There is no Lorentz transformation connecting two reference frames such that the field is purely magnetic in origin in one and purely electric in origin in the other.

ii. If $B^2 - E^2 > 0$ in a frame, then there exists a frame in which the field is purely magnetic.

iii. If $B^2 - E^2 < 0$ in a frame, then there exists a frame in which the field is purely electric.

iv. If $B^2 - E^2 = 0$ in a frame, then there exists a frame in which

— $\mathbf{B}$ is perpendicular to $\mathbf{E}$, if $\mathbf{B} \cdot \mathbf{E} = 0$.

— $\mathbf{B}$ is parallel to $\mathbf{E}$, if $\mathbf{B} \cdot \mathbf{E} > 0$.

— $\mathbf{B}$ is antiparallel to $\mathbf{E}$, if $\mathbf{B} \cdot \mathbf{E} < 0$.