Homework No. 06 (Spring 2015)

PHYS 520B: Electromagnetic Theory

Due date: Monday, 2015 Apr 20, 4.30pm

1. (**20 points.**) (Refer Schwinger et al. problem 10.11.)
   In covariant notation, the action for the electromagnetic field interacting with a prescribed current $j^\mu = (c\rho, \mathbf{j})$ is

$$W = \int d^4x \left[\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\mu_0} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + j^\mu A_\mu \right]. \tag{1}$$

   In the action the vector potential A_μ and the field strength tensor $F_{\mu\nu}$ are regarded as independent variables.

   (a) Derive

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

   and

$$\partial_\nu F^{\mu\nu} = \mu_0 \, j^\mu \tag{3}$$

   by requiring that W be stationary under independent variations in $F_{\mu\nu}$ and A_μ respectively.

   i. Further, derive the statement of conservation of charge,

$$\partial_\mu j^\mu = 0. \tag{4}$$

   ii. By adding the null term

$$\int d^4x \, \lambda \, \partial_\mu j^\mu \tag{5}$$

   to the action show that the action is invariant under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \lambda, \tag{6}$$

   where λ is an arbitrary function of spacetime.

   (b) Consider a general coordinate transformation

$$\bar{x}^\mu = x^\mu - \delta x^\mu. \tag{7}$$

   A scalar field $\phi(x)$ changes under such a transformation as

$$\delta\phi(x) = \phi(x - \delta x) - \phi(x) = -\delta x^\lambda \partial_\lambda \phi. \tag{8}$$

Because the action is invariant under a gauge transformation, we conclude that the vector potential A_μ responds to a general coordinate transformation as the derivative of a scalar field. Thus derive,

$$\delta A_\mu = -(\partial_\mu \delta x^\lambda) A_\lambda - \delta x^\lambda \partial_\lambda A_\mu. \tag{9}$$

Further, derive the response of the field strength tensor $F_{\mu\nu}$ to a general coordinate transformation as

$$\delta F_{\mu\nu} = -(\partial_\mu \delta x^\lambda) F_{\lambda\nu} - (\partial_\nu \delta x^\lambda) F_{\mu\lambda} - \delta x^\lambda \partial_\lambda F_{\mu\nu}. \tag{10}$$

(c) Now consider a source-free region, where $j^\mu = 0$, and the fields vanish outside the space-time region in question. Assume now that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, so that

$$W = -\int d^4x \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}. \tag{11}$$

Show that

$$\delta W = \int d^4x (\partial_\mu \delta x_\nu) t^{\mu\nu}, \tag{12}$$

where

$$t^{\mu\nu} = \frac{1}{\mu_0} F^{\mu\lambda} F^\nu{}_\lambda + g^{\mu\nu} \mathcal{L}. \tag{13}$$

(d) For $\delta x^\lambda =$ constant show that $\delta W = 0$.

(e) Use the action principle to show that $t^{\mu\nu}$ is conserved,

$$\partial_\mu t^{\mu\nu} = 0. \tag{14}$$

(f) Verify that t^{00} is the energy density,

$$t^{00} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 = U, \tag{15}$$

t^{i0} is the energy flux vector,

$$t^{i0} = \frac{1}{c}\mathbf{E} \times \mathbf{H} = \frac{1}{c}\mathbf{S}, \tag{16}$$

t^{0i} is the momentum density,

$$t^{0i} = c\mathbf{D} \times \mathbf{B} = c\mathbf{G}, \tag{17}$$

and t^{ij} is the momentum flux tensor,

$$\mathbf{t} = \mathbf{1} U - (\mathbf{DE} + \mathbf{BH}). \tag{18}$$

Thus, we have the conservation of energy

$$\frac{\partial U}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} = 0 \tag{19}$$

and conservation of momentum

$$\frac{\partial \mathbf{G}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{t} = 0. \tag{20}$$

(g) What is the trace of $t^{\mu\nu}$? What is the significance of that result?