Homework No. 01 (Spring 2015)

PHYS 530A: Quantum Mechanics II

Due date: Wednesday, 2015 Jan 28, 4.30pm

1. A relativistic charged particle of charge q and mass m in the presence of a known electric and magnetic field is described by

$$\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = q\mathbf{E} + \frac{q}{c}\mathbf{v}\times\mathbf{B}. \tag{1}$$

   (a) Find the Lagrangian for this system that implies the equation of motion of Eq. (1) using Hamilton's principle of stationary action.

   (b) Determine the canonical momentum for this system

   (c) Determine the Hamilton $H(\mathbf{p},\mathbf{r})$ for this system.

2. The Hamiltonian is defined by the relation

$$H(p_i, q_i, t) = \sum_i p_i\dot{q}_i - L(q_i, \dot{q}_i, t). \tag{2}$$

   Show that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{3}$$

   Under what circumstances is H interpreted as the energy of the system?

3. Consider the four-vector $x^\alpha = (ct, \mathbf{x})$. Here $\alpha = 0, 1, 2, 3$, such that $x^0 = ct$ and x^i are the three components of vector $\mathbf{x}$. The proper time s, that remains invariant under a Lorentz transformation, satisfies

$$-ds^2 = -c^2dt^2 + d\mathbf{x}\cdot d\mathbf{x}. \tag{4}$$

   Thus, derive the relation

$$\frac{1}{c}\frac{ds}{dt} = \sqrt{1-\frac{v^2}{c^2}}, \tag{5}$$

   where $\mathbf{v} = d\mathbf{x}/dt$. The energy E and momentum $\mathbf{p}$ of a particle of mass m is defined as

$$mc^2\frac{dx^\alpha}{ds} = (E, \mathbf{p}c). \tag{6}$$

   Find the explicit expressions for E and $\mathbf{p}$ in terms of $\mathbf{v}$, c, and m. Show that

$$\frac{dx^\alpha}{ds}\frac{dx_\alpha}{ds} = -1, \tag{7}$$

   and use this to derive $E^2 = p^2c^2 + m^2c^4$.

4. Consider the Lagrangian

$$L = \frac{1}{2} m \left(\frac{d\mathbf{r}}{dt} \right)^2 - V(\mathbf{r}, t). \tag{8}$$

(a) Show that principle of stationary action with respect to $\delta \mathbf{r}$ implies Newton's second law

$$m \frac{d^2 \mathbf{r}}{dt^2} = -\boldsymbol{\nabla} V. \tag{9}$$

(b) Show that principle of stationary action with respect to δt implies

$$\frac{d}{dt} \left[\frac{1}{2} m \left(\frac{d\mathbf{r}}{dt} \right)^2 + V \right] = \frac{\partial V}{\partial t}, \tag{10}$$

which for a static potential, $\partial V / \partial t = 0$, is the statement of conservation of energy.

(c) Show that the invariance of the total time derivative term, that gets contributions only from the end points, under an infinitesimal rigid rotation

$$\mathbf{r}' = \mathbf{r} - \delta \mathbf{r}, \qquad \delta \mathbf{r} = \delta \boldsymbol{\omega} \times \mathbf{r}, \tag{11}$$

implies the conservation of total angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

5. The Hamiltonian for a hydrogenic atom is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{Ze^2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{12}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the positions of the two constituent particles of masses m_1 and m_2 and charges e and Ze. Introduce the coordinates representing the center of mass, relative position, total momentum, and relative momentum:

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r_1} - \mathbf{r_2}, \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{p} = \frac{m_2 \mathbf{p}_1 - m_1 \mathbf{p}_2}{m_1 + m_2}, \tag{13}$$

respectively, to rewrite the Hamiltonian as

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} - \frac{Ze^2}{r}, \tag{14}$$

where

$$M = m_1 + m_2, \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{15}$$

Show that Hamilton's equations of motion are given by

$$\frac{d\mathbf{R}}{dt} = \frac{\mathbf{P}}{M}, \quad \frac{d\mathbf{P}}{dt} = 0, \quad \frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\mu}, \quad \frac{d\mathbf{p}}{dt} = -Ze^2 \frac{\mathbf{r}}{r^3}. \tag{16}$$

Verify that the Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the Laplace-Runge-Lenz vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{1}{\mu Z e^2} \mathbf{p} \times \mathbf{L}, \tag{17}$$

are the three constants of motion for a hydrogenic atom.