

Homework No. 02 (Spring 2015)

PHYS 530A: Quantum Mechanics II

Due date: Tuesday, 2015 Feb 10, 4.30pm

1. Show that the commutator of two matrices,

$$[\mathbf{A}, \mathbf{B}] \equiv \mathbf{AB} - \mathbf{BA}, \quad (1)$$

satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that

- (a) Antisymmetry:

$$[\mathbf{A}, \mathbf{B}] = -[\mathbf{B}, \mathbf{A}]. \quad (2)$$

- (b) Bilinearity: (a and b are numbers.)

$$[a\mathbf{A} + b\mathbf{B}, \mathbf{C}] = a[\mathbf{A}, \mathbf{C}] + b[\mathbf{B}, \mathbf{C}]. \quad (3)$$

Further show that

$$[\mathbf{AB}, \mathbf{C}] = \mathbf{A}[\mathbf{B}, \mathbf{C}] + [\mathbf{A}, \mathbf{C}]\mathbf{B}. \quad (4)$$

- (c) Jacobi's identity:

$$[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] + [\mathbf{B}, [\mathbf{C}, \mathbf{A}]] + [\mathbf{C}, [\mathbf{A}, \mathbf{B}]] = 0. \quad (5)$$

2. Show that the vector product of two vectors, in this problem denoted using

$$[\mathbf{A}, \mathbf{B}] \equiv \mathbf{A} \times \mathbf{B}, \quad (6)$$

satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that

- (a) Antisymmetry:

$$[\mathbf{A}, \mathbf{B}] = -[\mathbf{B}, \mathbf{A}]. \quad (7)$$

- (b) Bilinearity: (a and b are numbers.)

$$[a\mathbf{A} + b\mathbf{B}, \mathbf{C}] = a[\mathbf{A}, \mathbf{C}] + b[\mathbf{B}, \mathbf{C}]. \quad (8)$$

Further show that

$$[\mathbf{A} \times \mathbf{B}, \mathbf{C}] = \mathbf{A} \times [\mathbf{B}, \mathbf{C}] + [\mathbf{A}, \mathbf{C}] \times \mathbf{B}. \quad (9)$$

(c) Jacobi's identity:

$$[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] + [\mathbf{B}, [\mathbf{C}, \mathbf{A}]] + [\mathbf{C}, [\mathbf{A}, \mathbf{B}]] = 0. \quad (10)$$

3. Given F and G are constants of motion, that is

$$[F, H] = 0 \quad \text{and} \quad [G, H] = 0. \quad (11)$$

Then, using Jacobi's identity, show that $[F, G]$ is also a constant of motion. Thus, conclude the following:

- (a) If L_x and L_y are constants of motion, then L_z is also a constant of motion.
 - (b) If p_x and L_z are constants of motion, then p_y is also a constant of motion.
4. (Refer Goldstein, Sec. 9.5.) Hamiltonian for the motion of a ball (along the radial direction) near the surface of Earth is given by

$$H(z, p_z) = \frac{p_z^2}{2m} - mgz. \quad (12)$$

(a) Determine the equations of motions using

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_z} \quad \text{and} \quad \frac{dp_z}{dt} = -\frac{\partial H}{\partial z}. \quad (13)$$

Then, solve the coupled differential equations to find the familiar elementary solution

$$z = z_0 + \frac{p_0}{m}t + \frac{1}{2}gt^2. \quad (14)$$

(b) Next, determine the equations of motion using

$$[z, H] = \frac{\partial H}{\partial p_z} \quad \text{and} \quad [p_z, H] = -\frac{\partial H}{\partial z}. \quad (15)$$

Then, using

$$z = z_0 + t[z, H]_0 + \frac{1}{2}t^2[[z, H], H]_0 + \cdots \quad (16)$$

rederive the elementary solution. Here the subscript zero refers to the initial conditions at $t = 0$.