# Homework No. 05 (Spring 2015)

## PHYS 530A: Quantum Mechanics II

Due date: Thursday, 2015 Apr 1, 4.30pm

1. (**50 points.**) (Ref. Milton's notes.)

   (a) Consider three numerical vectors, **a**, **b**, **c**. Show that

   $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0. \tag{1}$$

   (b) Now consider operators $A$, $B$, $C$. Show that

   $$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{2}$$

   (c) The multiplication property of the Pauli spin matrices can be written as

   $$(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}). \tag{3}$$

   From this, show that

   $$\frac{1}{i\hbar}\left[\frac{\hbar}{2}\boldsymbol{\sigma} \cdot \mathbf{a}, \frac{\hbar}{2}\boldsymbol{\sigma} \cdot \mathbf{b}\right] = \frac{\hbar}{2}\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}). \tag{4}$$

   (d) More generally, what is

   $$\frac{1}{i\hbar}\left[\mathbf{J} \cdot \mathbf{a}, \mathbf{J} \cdot \mathbf{b}\right]? \tag{5}$$

   (e) Use

   $$A = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{a}, \qquad B = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{b}, \qquad \text{and} \qquad C = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{c} \tag{6}$$

   in the result of problem (1b) to derive the result of problem (1a).

2. (**20 points.**) (Ref. Milton's notes.) A vector operator **V** is defined by the transformation property

   $$\frac{1}{i\hbar}\left[\mathbf{V}, \delta\boldsymbol{\omega} \cdot \mathbf{J}\right] = \delta\boldsymbol{\omega} \times \mathbf{V}, \tag{7}$$

   which states the commutation relations of components of **V** with those of angular momentum **J**. Since a scalar operator $S$ does not change under rotations it is defined by the corresponding transformation

   $$\frac{1}{i\hbar}\left[S, \delta\boldsymbol{\omega} \cdot \mathbf{J}\right] = 0. \tag{8}$$

   (a) Show that the scalar product of vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ is a scalar.

(b) Show that the vector product of vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ is a vector.

3. (**50 points.**) For $j = 1$:

(a) Determine the matrix representantion for

$$J_z, J_x, J_y, J_+, J_-, \text{and } J^2. \tag{9}$$

(b) Evaluate

$$\mathrm{Tr}(J_k), \quad \mathrm{Tr}(J_k J_l), \quad \text{and} \quad \mathrm{Tr}(J_k^2 J_l^2), \quad \text{for} \quad k, l = x, y, z. \tag{10}$$