

Homework No. 08 (Spring 2015)

PHYS 530A: Quantum Mechanics II

Due date: Tuesday, 2015 Apr 28, 4.30pm

1. The polar equation of a conic of eccentricity ε is

$$r = \frac{a(1 - \varepsilon^2)}{1 - \varepsilon \cos \theta}, \quad (1)$$

where $2a$ is the major-axis of the ellipse. The directrix of an ellipse is a line perpendicular to the major-axis at a distance d from the focus (origin). For an ellipse, the ratio between the radial distance of a point on the ellipse from the origin, and the distance of the point from the directrix, is the eccentricity. Thus, determine d in terms of a and ε .

2. The components of the position and momentum operator, $\mathbf{r}$ and $\mathbf{p}$, respectively, satisfy the commutation relations $[r_i, p_j] = i\hbar\delta_{ij}$. Verify the following:

(a) $\mathbf{r} \times \mathbf{p} + \mathbf{p} \times \mathbf{r} = 0$.

(b) $\mathbf{r} \cdot \mathbf{p} - \mathbf{p} \cdot \mathbf{r} = 3i\hbar$.

(c) $(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{p}) - (\mathbf{a} \cdot \mathbf{p})(\mathbf{b} \cdot \mathbf{r}) = i\hbar(\mathbf{a} \cdot \mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ are numerical.

(d) $\mathbf{r} \times (\mathbf{r} \times \mathbf{p}) = \mathbf{r} \mathbf{p} \cdot \mathbf{r} - \mathbf{p} r^2 + i\hbar \mathbf{r}$.

3. Using commutation relations between $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{L}$, verify the following:

(a) $\mathbf{L} \times \mathbf{L} = i\hbar \mathbf{L}$.

(b) $\mathbf{p} \times \mathbf{L} + \mathbf{L} \times \mathbf{p} = 2i\hbar \mathbf{p}$.

(c) $-\mathbf{L} \times \mathbf{p} \cdot \frac{\mathbf{r}}{r} = L^2 \frac{1}{r} = \frac{1}{r} L^2$.

4. Verify the following equation of motion for the hydrogen atom,

$$\frac{1}{i\hbar}[\mathbf{p}, H] = -\frac{Ze^2 \mathbf{r}}{r^3}, \quad (2)$$

where the Hamiltonian is

$$H = \frac{p^2}{2\mu} - \frac{Ze^2}{r}. \quad (3)$$

5. Verify that the axial vector,

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{1}{\mu Ze^2} \frac{1}{2} (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}), \quad (4)$$

satisfies

$$\mathbf{A} \cdot \mathbf{L} = 0 \quad \text{and} \quad \mathbf{L} \cdot \mathbf{A} = 0. \quad (5)$$