Homework No. 05 (Spring 2016)

PHYS 530A: Quantum Mechanics II

Due date: Thursday, 2016 Mar 10, 4.30pm

1. (**20 points.**) Two matrices A and B satisfy the relation

$$AB - BA = 1. \tag{1}$$

   (a) Prove that this cannot be true in a finite dimensional vector space.
   Hint: Take trace.

   (b) Construct infinite dimensional matrices A and B that satisfy the above relation.
   Hint:

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2}$$

2. (**40 points.**) A unitary matrix is defined by

$$U^\dagger U = 1, \tag{3}$$

where $\dagger$ stands for transpose and complex conjugation.

   (a) Show that

$$U = e^{iH} \tag{4}$$

   is unitary if H is Hermitian, that is $H^\dagger = H$.

   (b) Show that

$$U = \frac{1 + iA}{1 - iA} \tag{5}$$

   is unitary if A is Hermitian.

   (c) Using

$$\tan^{-1} A = \frac{i}{2} \ln \left(\frac{1 - iA}{1 + iA} \right) \tag{6}$$

   show that

$$H = 2 \tan^{-1} A. \tag{7}$$

(d) Show that

$$U = \frac{1 - iB}{1 + iB} \tag{8}$$

is unitary if B is Hermitian.

3. (**20 points.**) Show that the combination $X^\dagger X$ is Hermitian, irrespective of X being Hermitian. Use this to deduce that the eigenvalues of $X^\dagger X$ are non-negative.

4. (**20 points.**) Prove that Hermitian operators have real eigenvalues. Further, show that any two eigenfunctions belonging to distinct (unequal) eigenvalues of a Hermitian operator are mutually orthogonal.

5. (**60 points.**) Let A and B be Hermitian operators.

(a) Consider the expectation or average values of A and B in the physical state $|\,\rangle$,

$$\langle A \rangle = \langle\,|A|\,\rangle, \qquad \langle B \rangle = \langle\,|B|\,\rangle, \tag{9}$$

and the mean square deviation from these averages,

$$(\delta A)^2 = \langle\,|(A - \langle A \rangle)^2|\,\rangle \equiv \langle 1|1 \rangle, \tag{10}$$

$$(\delta B)^2 = \langle\,|(B - \langle B \rangle)^2|\,\rangle \equiv \langle 2|2 \rangle, \tag{11}$$

where

$$\langle 1| = \langle\,|(A - \langle A \rangle), \qquad |2\rangle = (B - \langle B \rangle)|\,\rangle, \tag{12}$$

(b) (Prove the Schwarz inequality.) Use the Schwarz inequality to learn

$$(\delta A)^2 (\delta B)^2 = \langle 1|1 \rangle \langle 2|2 \rangle \geq |\langle 1|2 \rangle|^2, \tag{13}$$

where the equal sign applies only when $|1\rangle$ is parallel to $|2\rangle$.

(c) Show that the antisymmetric product of two Hermitian operators X and Y,

$$C = \frac{1}{i}(XY - YX) = \frac{1}{i}\big[X, Y\big], \tag{14}$$

is also Hermitian, that is, $C^\dagger = C$. Further, show that the symmetric construction,

$$(XY + YX) = \big\{X, Y\big\}, \tag{15}$$

is also Hermitian. Thus, the product XY, which is not Hermitian, can be expressed as a combination of two Hermitian operators,

$$XY = \frac{1}{2}(XY + YX) + \frac{i}{2}C. \tag{16}$$

Remember that the expectation values of Hermitian operators are real.

(d) Let

$$X = A - \langle A \rangle, \qquad Y = B - \langle B \rangle. \tag{17}$$

Thus, derive

$$|\langle\ |(XY)|\ \rangle|^2 = \frac{1}{4}|\langle\ |(XY + YX)|\ \rangle|^2 + \frac{1}{4}|\langle\ |C|\ \rangle|^2. \tag{18}$$

(e) Using Eq. (18) in Eq. (13) derive Robertson's generalization of Heisenberg's uncertainty relation

$$(\delta A)(\delta B) \geq \frac{1}{2}|\langle C \rangle|. \tag{19}$$

(f) Apply this to the pairs $(A, B) = (q, p)$ and $(A, B) = (\sigma_x, \sigma_y)$.