Homework No. 01 (Fall 2016)

PHYS 530B: Quantum Mechanics II

Due date: Thursday, 2016 Sep 1, 4.30pm

1. (**20 points.**) The components of the position and momentum operator, $\mathbf{r}$ and $\mathbf{p}$, respectively, satisfy the commutation relations $[r_i, p_j] = i\hbar\delta_{ij}$. Verify the following:

   (a) $\mathbf{r}\times\mathbf{p} + \mathbf{p}\times\mathbf{r} = 0$.

   (b) $\mathbf{r}\cdot\mathbf{p} - \mathbf{p}\cdot\mathbf{r} = 3i\hbar$.

   (c) $(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{p}) - (\mathbf{b}\cdot\mathbf{p})(\mathbf{a}\cdot\mathbf{r}) = i\hbar(\mathbf{a}\cdot\mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ and numerical.

   (d) $\mathbf{r}\times(\mathbf{r}\times\mathbf{p}) = \mathbf{r}\,\mathbf{p}\cdot\mathbf{r} - \mathbf{p}r^2 + i\hbar\mathbf{r}$.

2. (**20 points.**) Using commutation relations between $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{L}$, verify the following:

   (a) $\mathbf{L}\times\mathbf{L} = i\hbar\mathbf{L}$.

   (b) $\mathbf{p}\times\mathbf{L} + \mathbf{L}\times\mathbf{p} = 2i\hbar\mathbf{p}$.

   (c) $-\mathbf{L}\times\mathbf{p}\cdot\dfrac{\mathbf{r}}{r} = L^2\dfrac{1}{r} = \dfrac{1}{r}L^2$.

   (d) $\mathbf{p}\times\mathbf{L}\cdot\mathbf{p} = 2i\hbar\,p^2$.

3. (**20 points.**) Verify that the axial vector,

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{1}{\mu Z e^2}\frac{1}{2}(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}), \tag{1}$$

   satisfies

$$\mathbf{A}\cdot\mathbf{L} = 0 \qquad \text{and} \qquad \mathbf{L}\cdot\mathbf{A} = 0. \tag{2}$$

4. (**30 points.**) Using commutation relations between $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{L}$, verify the relation

$$\mathbf{p}\times\mathbf{L}\cdot\mathbf{p} = 2i\hbar\,p^2. \tag{3}$$

   Thus, verify that either of the equalities for

$$\mathbf{M} = -\frac{1}{2}\left(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}\right) = -\mathbf{p}\times\mathbf{L} + i\hbar\mathbf{p} = \mathbf{L}\times\mathbf{p} - i\hbar\mathbf{p} \tag{4}$$

   leads to

$$M^2 = (L^2 + \hbar^2)p^2. \tag{5}$$

Comment: This ensures that either of the expressions for the Axial vector

$$\mathbf{A} = \hat{\mathbf{r}} - \frac{1}{\mu Z e^2}\frac{1}{2}\left(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}\right) \tag{6a}$$

$$= \hat{\mathbf{r}} - \frac{1}{\mu Z e^2}\mathbf{p}\times\mathbf{L} + \frac{i\hbar}{\mu Z e^2}\mathbf{p} \tag{6b}$$

$$= \hat{\mathbf{r}} + \frac{1}{\mu Z e^2}\mathbf{L}\times\mathbf{p} - \frac{i\hbar}{\mu Z e^2}\mathbf{p} \tag{6c}$$

leads to

$$A^2 = 1 + \frac{2(L^2 + \hbar^2)H}{\mu Z^2 e^4}. \tag{7}$$