Homework No. 03 (Fall 2016)

PHYS 530B: Quantum Mechanics II

Due date: Monday, 2016 Sep 26, 4.30pm

1. (**75 points.**) In this problem we shall construct the ground state of a hydrogenic atom, $|100\rangle$. For the ground state, $n = 1$, we have $j_1 = 0$, $j_2 = 0$, which corresponds to $l = 0$, $m = 0$. Thus, we conclude

$$\mathbf{J}^{(1)}|100\rangle = 0 \qquad \text{and} \qquad \mathbf{J}^{(2)}|100\rangle = 0, \tag{1}$$

which implies

$$\mathbf{L}|100\rangle = 0 \qquad \text{and} \qquad \mathbf{A}|100\rangle = 0. \tag{2}$$

Show that, in conjunction, these equations imply

$$\left(\frac{\mathbf{r}}{r} + \frac{i\hbar\,\mathbf{p}}{\mu Z e^2}\right)|100\rangle = 0. \tag{3}$$

(a) With the goal of finding the projection of the ground state in the position basis, the wavefunction

$$\langle \mathbf{r}|100\rangle = \psi_{100}(\mathbf{r}), \tag{4}$$

we identify the Bohr radius

$$a_0 = \frac{\hbar^2}{\mu e^2}, \tag{5}$$

to rewrite Eq. (3) in the form

$$\left(\frac{\mathbf{r}}{r} + \frac{a_0}{Z}\frac{i}{\hbar}\mathbf{p}\right)|100\rangle = 0. \tag{6}$$

Show that the projection on the position basis leads to the differential equation for the wavefunction

$$\left(\frac{\mathbf{r}}{r} + \frac{a_0}{Z}\boldsymbol{\nabla}\right)\psi_{100}(\mathbf{r}) = 0. \tag{7}$$

Thus, determine the normalized ground state wavefunction to be

$$\psi_{100}(\mathbf{r}) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{\frac{3}{2}} e^{-Z\frac{r}{a_0}}. \tag{8}$$

(b) With the goal of finding the projection of the ground state in the momentum basis,

$$\langle \mathbf{p}|100\rangle = \psi_{100}(\mathbf{p}), \tag{9}$$

we identify the wavevector corresponding to the Bohr radius, $k_0 = 2\pi/a_0$, and the related Bohr momentum

$$p_0 = \frac{\hbar}{a_0}, \tag{10}$$

to rewrite Eq. (3) in the form

$$\left(\frac{\mathbf{r}}{r} + \frac{i}{Z}\frac{\mathbf{p}}{p_0}\right)|100\rangle = 0. \tag{11}$$

Use the Hamiltonian to interpret

$$\frac{1}{r}|100\rangle = \frac{(p^2 + Z^2 p_0^2)}{2\mu Z e^2}|100\rangle, \tag{12}$$

and show that

$$\Big[\mathbf{r}(p^2 + Z^2 p_0^2) + 2i\hbar\mathbf{p}\Big]|100\rangle = 0. \tag{13}$$

Show that the projection on the momentum basis leads to the differential equation

$$\left(\frac{\partial}{\partial\mathbf{p}} + \frac{4\mathbf{p}}{(p^2 + Z^2 p_0^2)}\right)\psi_{100}(\mathbf{p}) = 0. \tag{14}$$

Thus, determine the solution

$$\psi_{100}(\mathbf{p}) = \frac{2}{\pi}\frac{\sqrt{2Z^5 p_0^5}}{(p^2 + Z^2 p_0^2)^2}. \tag{15}$$

(c) Evaluate the Fourier transformation

$$\psi_{100}(\mathbf{p}) = \hbar^{-\frac{3}{2}}\int d^3\mathbf{r}\, e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}}\,\psi_{100}(\mathbf{r}), \tag{16}$$

and verify that this is indeed equal to the result in Eq. (15).

2. (**20 points.**) Derive the relation

$$\mathbf{r}\,H_0 = -\frac{3Ze^2}{4}\mathbf{A} + \frac{d\mathbf{X}}{dt}, \tag{17}$$

where

$$\mathbf{X} = -\frac{1}{8}(\mathbf{r}\times\mathbf{L} - \mathbf{L}\times\mathbf{r}) + \frac{3}{4}i\hbar\,\mathbf{r}. \tag{18}$$

Take the time average and give the physical interpretation of this equation in classical mechanics.