Homework No. 06 (Fall 2016)

PHYS 530B: Quantum Mechanics II

Due date: Monday, 2016 Nov 17, 4.30pm

1. (**20 points.**) A free particle is described by the Hamiltonian
$$H = \boldsymbol{\alpha} \cdot \mathbf{p} c + \beta m c^2. \tag{1}$$
Starting from the above equation derive the Lorentz covariant form of Dirac equation
$$\boldsymbol{\gamma}^\mu p_\mu + mc = 0, \tag{2}$$
where $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma}) = (\beta, \beta\boldsymbol{\alpha})$ and $p^\mu = (E/c, \mathbf{p})$.

2. (**40 points.**) A free particle is described by the Hamiltonian
$$H = \boldsymbol{\alpha} \cdot \mathbf{p} c + \beta m c^2. \tag{3}$$
Derive the following equations of motion:
$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\alpha} c, \tag{4a}$$
$$\frac{d\mathbf{p}}{dt} = 0, \tag{4b}$$
$$\frac{i\hbar}{2}\frac{d\boldsymbol{\alpha}}{dt} = -i\boldsymbol{\Sigma} \times \mathbf{p} c + \boldsymbol{\alpha}\beta m c^2, \tag{4c}$$
$$\frac{i\hbar}{2}\frac{d\beta}{dt} = \beta\boldsymbol{\alpha} \cdot \mathbf{p} c. \tag{4d}$$
Derive the relations
$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\alpha} \times \mathbf{p} c, \qquad \text{where} \qquad \mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{5a}$$
$$\frac{d}{dt}\left(\frac{\hbar}{2}\boldsymbol{\Sigma}\right) = -\boldsymbol{\alpha} \times \mathbf{p} c, \qquad \text{where} \qquad \boldsymbol{\Sigma} = -\frac{i}{2}\boldsymbol{\alpha} \times \boldsymbol{\alpha}, \tag{5b}$$
$$\frac{d\mathbf{J}}{dt} = 0, \qquad \text{where} \qquad \mathbf{J} = \mathbf{L} + \frac{\hbar}{2}\boldsymbol{\Sigma}, \tag{5c}$$
which unambiguously indicates the existance of the Spin. Further, deduce
$$\frac{d}{dt}(\boldsymbol{\Sigma} \cdot \mathbf{p}) = 0, \tag{6a}$$
$$\frac{d}{dt}\left(\frac{\hbar}{2}\boldsymbol{\Sigma} \cdot \mathbf{L}\right) = -c\boldsymbol{\alpha} \cdot (\mathbf{p} \times \mathbf{L}) + i\hbar c\boldsymbol{\alpha} \cdot \mathbf{p}, \tag{6b}$$
$$\frac{d\mathbf{K}}{dt} = 0, \qquad \text{where} \qquad K = \beta(\boldsymbol{\Sigma} \cdot \mathbf{L} + \hbar). \tag{6c}$$

3. (**40 points.**)

(a) The momentum of a non-relativistic classical (non-quantum) free particle of mass m is related to it's velocity by the relation

$$\mathbf{v} = \frac{\mathbf{p}}{m}. \tag{7}$$

(b) Show that the momentum of a relativistic classical (non-quantum) free particle of mass m is related to it's velocity by the relation

$$\mathbf{v} = \frac{\mathbf{p}c}{E/c}, \tag{8}$$

where $E^2 = p^2c^2 + m^2c^4$.

(c) Let us denote a time derivative with a dot over the respective variable. The 'velocity' of relativistic quantum free particle of mass m is $\boldsymbol{\alpha}c$, as per the equation of motion

$$\dot{\mathbf{r}} = \boldsymbol{\alpha}c. \tag{9}$$

Derive the equation of motion

$$i\hbar\dot{\boldsymbol{\alpha}} = 2\mathbf{p}c - 2H\boldsymbol{\alpha} = -2\mathbf{p}c + 2\boldsymbol{\alpha}H. \tag{10}$$

(d) Derive the equation of motion

$$i\hbar\ddot{\boldsymbol{\alpha}} = -2H\dot{\boldsymbol{\alpha}} = 2\dot{\boldsymbol{\alpha}}H. \tag{11}$$

(e) Integrate to yield

$$\dot{\boldsymbol{\alpha}}(t) = e^{i\frac{2H}{\hbar}t}\dot{\boldsymbol{\alpha}}(0) = \dot{\boldsymbol{\alpha}}(0)e^{-i\frac{2H}{\hbar}t}, \tag{12}$$

where $\dot{\boldsymbol{\alpha}}(0)$ is a constant and thus commutes with the Hamiltonian. Verify this solution by differentiating it with time. The ordering of H matters because the Hamiltonian does not commute with $\dot{\boldsymbol{\alpha}}(t)$.

(f) Integrate again to derive the relation between momentum and 'velocity',

$$c\boldsymbol{\alpha}(t) = \frac{\mathbf{p}c}{H/c} + \frac{i\hbar c}{2}\dot{\boldsymbol{\alpha}}(0)e^{-i\frac{2H}{\hbar}t}H^{-1} \tag{13a}$$

$$= \frac{\mathbf{p}c}{H/c} + \frac{i\hbar c}{2}H^{-1}\dot{\boldsymbol{\alpha}}(0)e^{i\frac{2H}{\hbar}t}. \tag{13b}$$

The frist term relates to the classical relativistic formula. The second term oscillates with a frequency of $2H/\hbar$, which is at least $2mc^2/\hbar$. Evaluate this frequency, for an electron, in units of radian/second. This oscillatory part of the motion is called Zitterbewegung. For comparison, the plasma frequency associated with oscillations in electron density inside a conductor is about 10^{16} radian/second.