Final Exam (Fall 2017)

PHYS 440: Quantum Mechanics

Date: 2017 Dec 12

1. (**20 points.**) The Pauli matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

is written in the eigenbasis of

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Write σ_x in the eigenbasis of

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3}$$

2. (**20 points.**) Let us construct the total angular momentum states for the composite system built out of two angular momenta $j_1 = 1, j_2 = \frac{1}{2}$. Beginning with the state

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = |1, 1\rangle_{①} \left|\frac{1}{2}, \frac{1}{2}\right\rangle_{②} \tag{4}$$

use the lowering operation

$$J_-|j, m\rangle = \hbar\sqrt{(j+m)(j-m+1)}|j, m-1\rangle \tag{5}$$

to build the state $|3/2, -1/2\rangle$.

3. (**20 points.**) Evaluate the commutation relation

$$\frac{1}{i\hbar}[\mathbf{p}, H] \tag{6}$$

when the Hamiltonian is that of the hydrogen atom,

$$H = \frac{p^2}{2\mu} - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}. \tag{7}$$

This leads to the equation of motion for the hydrogen atom.

4. (**20 points.**) Using commutation relations between $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{L}$, verify the following:

$$\mathbf{p} \times \mathbf{L} + \mathbf{L} \times \mathbf{p} = 2i\hbar\mathbf{p}. \tag{8}$$

5. (**20 points.**) Consider a scattering process that involves an incident plane wave of energy $E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$, moving in the positive z direction, interacting with the potential of an inverted finite spherical well of radius a

$$V(\mathbf{r}') = \begin{cases} V, & r' < a, \\ 0, & r' > a. \end{cases} \tag{9}$$

The leading order contribution to the scattering amplitude, for this process, in the eikonal approximation (small angle large momentum) is

$$f^{(0)}(\theta) = \frac{k}{i} \int_0^\infty b db\, J_0(kb\theta) \left[e^{i\chi(b)} - 1 \right], \tag{10}$$

where

$$\chi(b) = -\frac{k}{2E} \int_{-\infty}^\infty dz'\, V(b, z'). \tag{11}$$

Here b is the magnitude of the projection of the coordinate $\mathbf{r}'$ in the plane perpendicular to the direction of z,

$$\mathbf{r}' = b \cos\phi'\, \hat{\mathbf{i}} + b \sin\phi'\, \hat{\mathbf{j}} + z'\, \hat{\mathbf{k}}, \tag{12}$$

and the coordinate $\hat{\mathbf{r}}$ represents the position of the detector,

$$\hat{\mathbf{r}} = \sin\theta \cos\phi\, \hat{\mathbf{i}} + \sin\theta \sin\phi\, \hat{\mathbf{j}} + \cos\theta\, \hat{\mathbf{k}}. \tag{13}$$

The process is independent of ϕ and ϕ' because of the azimuthal symmetry in the potential. Evaluate $\chi(b)$ for the process.