Midterm Exam No. 01 (Fall 2017)

PHYS 440: Quantum Mechanics

Date: 2017 Sep 19

1. (**25 points.**) The Pauli matrix
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1}$$
is written in the eigenbasis of σ_z. Write σ_y in the eigenbasis of σ_x.

2. (**25 points.**) The eigenfunctions for the Stern-Gerlach experiment are
$$\psi_+(\theta,\phi) = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix} \quad \text{and} \quad \psi_-(\theta,\phi) = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix}, \tag{2}$$
up to a phase. Using the notation for the probability for a measurement in the Stern-Gerlach experiment, introduced in the class, calculate
$$p\big([+;0,0] \to [+;\frac{\pi}{2},0] \to [+;0,0]\big). \tag{3}$$

3. (**25 points.**) Consider the rotation matrix
$$\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{4}$$
   (a) Find the eigenvalues of the matrix $\mathbf{A}$.
   (b) Find the normalized eigenvectors of matrix $\mathbf{A}$. (Simplification is achieved by writing the trignometric functions in terms of half angles. $1-\cos\theta = 2\sin^2\theta/2$, $1+\cos\theta = 2\cos^2\theta/2$, $\sin\theta = 2\sin\theta/2\cos\theta/2$.)
   (c) Determine the matrix that diagonalizes the matrix $\mathbf{A}$.
   (d) What can you then conclude about the eigenvalues and eigenvectors of $\mathbf{A}^{107}$? Find them.

4. (**25 points.**) Two matrices A and B satisfy the relation
$$AB - BA = 1. \tag{5}$$
   (a) Prove that this cannot be true in a finite dimensional vector space.
   Hint: Take trace.

(b) Nevertheless, construct infinite dimensional matrices A and B that satisfy the above relation.
Hint: The answer is not unique. For example, try

$$A = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{6}$$

5. (**100 points.**) Consider a normalized state

$$|\,\rangle = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x_1 + ix_2 \\ y_1 + iy_2 \end{pmatrix}, \qquad |u|^2 + |v|^2 = 1. \tag{7}$$

(a) Show that the expectation value of the Pauli matrices with respect to the above state satisfies

$$\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 = 1. \tag{8}$$

Thus, conclude that the magnitude of the expectation value of each of the Pauli matrix is less than or equal to unity.

(b) Define the errors in the measurement of the Pauli matrices to be

$$(\delta\sigma'_x)^2 = \langle\,|\big\{\sigma_x - \langle\sigma_x\rangle\big\}^2|\,\rangle, \tag{9a}$$
$$(\delta\sigma'_y)^2 = \langle\,|\big\{\sigma_y - \langle\sigma_y\rangle\big\}^2|\,\rangle, \tag{9b}$$
$$(\delta\sigma'_z)^2 = \langle\,|\big\{\sigma_z - \langle\sigma_z\rangle\big\}^2|\,\rangle. \tag{9c}$$

Show that

$$(\delta\sigma'_x)^2 = 1 - \langle\sigma_x\rangle^2, \tag{10a}$$
$$(\delta\sigma'_y)^2 = 1 - \langle\sigma_y\rangle^2, \tag{10b}$$
$$(\delta\sigma'_z)^2 = 1 - \langle\sigma_z\rangle^2. \tag{10c}$$

Thus, conclude that the errors in the measurement of each of the Pauli matrix is less than or equal to unity. Show that

$$(\delta\sigma'_x)^2 + (\delta\sigma'_y)^2 + (\delta\sigma'_z)^2 = 2. \tag{11}$$

(c) Using Robertson's generalization of Heisenberg's uncertainty relation

$$(\delta A)(\delta B) \geq \frac{1}{2}|\langle C\rangle|, \tag{12}$$

deduce the uncertainty relations for the Pauli matrices to be

$$(\delta\sigma'_x)^2(\delta\sigma'_y)^2 \geq \langle\sigma_z\rangle^2, \tag{13a}$$
$$(\delta\sigma'_y)^2(\delta\sigma'_z)^2 \geq \langle\sigma_x\rangle^2, \tag{13b}$$
$$(\delta\sigma'_z)^2(\delta\sigma'_x)^2 \geq \langle\sigma_y\rangle^2. \tag{13c}$$

Combine these uncertainty relations to derive an uncertainty relation involving all the three Pauli matrices,

$$(\delta\sigma'_x)^2(\delta\sigma'_y)^2 + (\delta\sigma'_y)^2(\delta\sigma'_z)^2 + (\delta\sigma'_z)^2(\delta\sigma'_x)^2 \geq 1. \tag{14}$$

(d) Show that minimal uncertainty in each of the above relations is attained, if

$$\langle\sigma_x\rangle\langle\sigma_y\rangle = 0, \tag{15a}$$
$$\langle\sigma_y\rangle\langle\sigma_z\rangle = 0, \tag{15b}$$
$$\langle\sigma_z\rangle\langle\sigma_x\rangle = 0. \tag{15c}$$

Since the expectation value of the Pauli matrices satisfy Eq. (8) all three of the expectation value of Pauli matrices can not be zero for a given state. But, two of them can be zero, which is sufficient for minimizing all the above uncertainty relations. For example, a state satisfying

$$\langle\sigma_x\rangle = \langle\sigma_y\rangle = 0 \tag{16}$$

minimizes all the four uncertainty relations above.

(e) Show that

$$|\text{min}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{17}$$

is such a state. Evaluate $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$, $\langle\sigma_z\rangle$, $\langle\sigma_x^2\rangle$, $\langle\sigma_y^2\rangle$, $\langle\sigma_z^2\rangle$, $\delta\sigma'_x$, $\delta\sigma'_y$, and $\delta\sigma'_z$, when the system is in this state.

(f) Understand the Stern-Gerlach experiment in this light.