Midterm Exam No. 01 (Spring 2018)

PHYS 530A: Quantum Mechanics II

Date: 2018 Feb 22

1. (**20 points.**) The Pauli matrix
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$
is written in the eigenbasis of
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$
Write σ_x in the eigenbasis of
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3}$$
Note that your result will have the arbitraryness of the choice of phase.

2. (**20 points.**) Using the notation for the probability for a measurement in the Stern-Gerlach experiment, introduced in the class, evaluate
$$p\left([+;0,0] \to [+;\frac{\pi}{2},0] \to [+;\frac{\pi}{2},\frac{\pi}{2}] \to [+;0,0]\right). \tag{4}$$
For reference, the eigenvectors for the Stern-Gerlach Hamiltonian are
$$|+;\theta,\phi\rangle = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix} \quad \text{and} \quad |-;\theta,\phi\rangle = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix}. \tag{5}$$

3. (**20 points.**) Consider the rotation matrix
$$\mathbf{A} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}. \tag{6}$$
   (a) Find the eigenvalues of the matrix $\mathbf{A}$.
   (b) Find the normalized eigenvectors of matrix $\mathbf{A}$.
   (c) Determine the matrix that diagonalizes the matrix $\mathbf{A}$.
   (d) What can you then conclude about the eigenvalues and eigenvectors of $\ln \mathbf{A}$? Find them.

4. (**20 points.**) Consider the operator construction

$$K(\lambda) = e^{-\lambda A} B e^{\lambda A} \tag{7}$$

in terms of two operators A and B, λ being a number. Show that

$$\frac{\partial K}{\partial \lambda} = [K, A]. \tag{8}$$

Evaluate the higher derivatives

$$\frac{\partial^n K}{\partial \lambda^n} \tag{9}$$

recursively. Thus, using Taylor expansion around $\lambda = 0$, show that

$$K(\lambda) = B + \lambda [B, A] + \frac{\lambda^2}{2!}[[B, A], A] + \frac{\lambda^3}{3!}[[[B, A], A], A] + \dots. \tag{10}$$

Then, for $\lambda = 1$, we have

$$e^{-A} B e^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] + \dots. \tag{11}$$

This is the Baker-Campbell-Hausdorff formula. Using this evaluate

$$U(\phi) = e^{-\frac{i}{2}\phi\sigma_z} \sigma_x e^{\frac{i}{2}\phi\sigma_z}, \tag{12}$$

where σ_x, σ_y, and σ_z, are Pauli matrices and ϕ is a number representing an angle of rotation. In particular, express your answer in the form

$$U(\phi) = a_0(\phi) + \boldsymbol{\sigma} \cdot \mathbf{a}(\phi) \tag{13}$$

and determine $a_0(\phi)$ and $\mathbf{a}(\phi)$.