Midterm Exam No. 02 (Spring 2018)

PHYS 530A: Quantum Mechanics II

Date: 2018 Apr 5

1. (**20 points.**) The probabilities in a series setup of Stern-Gerlach experiment can be described using the notation

$$p([A = a'] \to [B = b'] \to [C = c']), \tag{1}$$

where $[A = a']$ denotes the selection of the beam corresponding to the eigenvalue a'. For spin-$\frac{1}{2}$ the operators A, B, and C, are given by $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the direction of the magnetic field. Connecting with the notation introduced in class, we have, for example, $[\sigma_x = +1] = [+; \theta = \frac{\pi}{2}, \phi = 0]$.

   (a) Find the following probabilities:

$$p([\sigma_x = +1] \to [\sigma_x = +1]), \tag{2a}$$

$$p([\sigma_x = +1] \to [\sigma_y = +1] \to [\sigma_x = +1]) \tag{2b}$$

   Does the measurement of σ_y *completely* wipe out the prior knowledge of the measurement of σ_x? If yes, why? If no, why not? Are σ_x and σ_y complementary?

   (b) An experiment is capable of measuring the following six physical variables:

$$J_x, J_y, J_z, J_x^2, J_y^2, J_z^2, \tag{3}$$

   where

$$\mathbf{J} = \frac{\hbar}{2}\boldsymbol{\sigma}. \tag{4}$$

   Out of the 15 distinct pairs of variables above, list the pairs that can be measured simultaneously. That is, list those for which the measurement of one variable in a pair does not disturb the measurement of the other–the measurements are compatible.

   (c) Are the remaining pairs in the list complementary sets? Remember, complementary variables have optimal incompatibility. Complementary variables are pairs that are needed to describe the system, but the measurement of one variable completely (and maximally) wipes out the prior knowledge of the other.

2. (**20 points.**) A quantum harmonic oscillator can be constructed out of two non-Hermitian operators, y and $y^\dagger$, that satisfy the commutation relation

$$[y, y^\dagger] = 1. \tag{5}$$

The eigenstate spectrum of the (Hermitian) number operator, $n = y^\dagger y$, represented by $|n'\rangle$, where $n' = 0, 1, 2, \ldots$, satisfy

$$n|n'\rangle = n'|n'\rangle, \qquad y|n'\rangle = \sqrt{n'}|n'-1\rangle, \qquad y^\dagger|n'\rangle = \sqrt{n'+1}|n'+1\rangle. \tag{6}$$

(a) Build the matrix representation of the lowering operator y using

$$y = \begin{bmatrix} \langle 0|y|0\rangle & \langle 0|y|1\rangle & \langle 0|y|2\rangle & \langle 0|y|3\rangle & \langle 0|y|4\rangle & \cdots \\ \langle 1|y|0\rangle & \langle 1|y|1\rangle & \langle 1|y|2\rangle & \langle 1|y|3\rangle & \langle 1|y|4\rangle & \cdots \\ \langle 2|y|0\rangle & \langle 2|y|1\rangle & \langle 2|y|2\rangle & \langle 2|y|3\rangle & \langle 2|y|4\rangle & \cdots \\ \langle 3|y|0\rangle & \langle 3|y|1\rangle & \langle 3|y|2\rangle & \langle 3|y|3\rangle & \langle 3|y|4\rangle & \cdots \\ \langle 4|y|0\rangle & \langle 4|y|1\rangle & \langle 4|y|2\rangle & \langle 4|y|3\rangle & \langle 4|y|4\rangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{7}$$

Kindly calculate the first 5×5 block of the infinite dimensional matrix to report the pattern in the following questions.

(b) Similarly, build the matrix representation of the raising operator $y^\dagger$.

(c) Build the matrix representation of the number operator n.

(d) Using the constructions

$$y = \frac{1}{\sqrt{2\hbar}}(x + ip) \quad \text{and} \quad y^\dagger = \frac{1}{\sqrt{2\hbar}}(x - ip), \tag{8}$$

determine the matrix representations for the Hermitian operators, x and p. Check that x and p are indeed Hermitian matrices.

(e) Determine the matrices for the operators xp and px, and verify the commutation relation

$$\frac{1}{i\hbar}[x, p] = 1. \tag{9}$$

3. (**20 points.**) Given

$$\gamma = \sigma_x + \epsilon\sigma_z. \tag{10}$$

Find the probablity

$$p([\sigma_x = +1] \to [\gamma = +] \to [\sigma_x = +1]). \tag{11}$$

4. (**20 points.**) A vector operator $\mathbf{V}$ is defined by the transformation property

$$\frac{1}{i\hbar}\,[\mathbf{V}, \delta\boldsymbol{\omega}\cdot\mathbf{J}] = \delta\boldsymbol{\omega}\times\mathbf{V}, \tag{12}$$

which states the commutation relations of components of $\mathbf{V}$ with those of angular momentum $\mathbf{J}$. Since a scalar operator S does not change under rotations it is defined by the corresponding transformation

$$\frac{1}{i\hbar}\,[S, \delta\boldsymbol{\omega}\cdot\mathbf{J}] = 0. \tag{13}$$

Using Eq. (12), show that the scalar product of two vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ is a scalar. That is,

$$\frac{1}{i\hbar}\,[\mathbf{V}_1\cdot\mathbf{V}_2, \delta\boldsymbol{\omega}\cdot\mathbf{J}] = 0. \tag{14}$$