Homework No. 01 (Spring 2018)

PHYS 530A: Quantum Mechanics II

Due date: Tuesday, 2018 Jan 23, 4.30pm

1. (**20 points.**) A 3×3 matrix A satisfies the equation
$$A^3 = 1. \tag{1}$$
Given that the eigenvalues of A are non-degenerate, find all eigenvalues of A.

2. (**20 points.**) Consider the rotation matrix
$$\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2}$$
   (a) Find the eigenvalues of the matrix $\mathbf{A}$.
   (b) Find the normalized eigenvectors of matrix $\mathbf{A}$.
   (c) Determine the matrix that diagonalizes the matrix $\mathbf{A}$.
   (d) What can you then conclude about the eigenvalues and eigenvectors of $\mathbf{A}^{107}$? Find them.

3. (**20 points.**) Two matrices A and B satisfy the relation
$$AB - BA = 1. \tag{3}$$
   (a) Prove that this cannot be true in a finite dimensional vector space.
   Hint: Take trace.
   (b) Nevertheless, construct infinite dimensional matrices A and B that satisfy the above relation.
   Hint: The answer is not unique. For example,
$$A = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{4}$$

Construct another example with

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{5}$$

These examples exploit the counterintuitive properties of divergent series.