Homework No. 02 (Spring 2018)

PHYS 530A: Quantum Mechanics II

Due date: Tuesday, 2018 Jan 30, 4.30pm

1. (**20 points.**) The Pauli matrices satisfy

$$\sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k. \tag{1}$$

Thus, evaluate the commutation relations

$$\left[\sigma_x^n, \sigma_y^m\right] \tag{2}$$

for positive integers n and m.

2. (**40 points.**) Consider the eigenvalue equation

$$\sigma_x |\sigma_x'\rangle = \sigma_x' |\sigma_x'\rangle, \tag{3}$$

where primes denote eigenvalues.

(a) Find the eigenvalues and normalized eigenvectors (up to a phase) of

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

For reference we shall call these eigenvectors $|\sigma_x' = +\rangle$ and $|\sigma_x' = -\rangle$.

(b) Now compute the new matrix

$$\bar{\sigma}_x = \begin{pmatrix} \langle \sigma_x' = +|\sigma_x|\sigma_x' = +\rangle & \langle \sigma_x' = +|\sigma_x|\sigma_x' = -\rangle \\ \langle \sigma_x' = -|\sigma_x|\sigma_x' = +\rangle & \langle \sigma_x' = -|\sigma_x|\sigma_x' = -\rangle \end{pmatrix}. \tag{5}$$

(c) Similarly, compute the new matrices

$$\bar{\sigma}_y = \begin{pmatrix} \langle \sigma_x' = +|\sigma_y|\sigma_x' = +\rangle & \langle \sigma_x' = +|\sigma_y|\sigma_x' = -\rangle \\ \langle \sigma_x' = -|\sigma_y|\sigma_x' = +\rangle & \langle \sigma_x' = -|\sigma_y|\sigma_x' = -\rangle \end{pmatrix}, \quad \text{where} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{6}$$

and

$$\bar{\sigma}_z = \begin{pmatrix} \langle \sigma_x' = +|\sigma_z|\sigma_x' = +\rangle & \langle \sigma_x' = +|\sigma_z|\sigma_x' = -\rangle \\ \langle \sigma_x' = -|\sigma_z|\sigma_x' = +\rangle & \langle \sigma_x' = -|\sigma_z|\sigma_x' = -\rangle \end{pmatrix}, \quad \text{where} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

(d) Find the product of the last two matrices, $\bar{\sigma}_y \bar{\sigma}_z$, and express it in terms of $\bar{\sigma}_x$.

3. (**20 points.**) The Pauli matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8}$$

is written in the eigenbasis of

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

Write σ_x in the eigenbasis of

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{10}$$

Note that this representation has the arbitraryness of the choice of phase in the eigenvectors.

4. (**20 points.**) If $\boldsymbol{\sigma}$ is the vector constructed out of Pauli matrices and $\mathbf{a}$ is a (numerical) vector, evaluate

$$\mathrm{tr}\cos(\boldsymbol{\sigma} \cdot \mathbf{a}). \tag{11}$$