Homework No. 03 (Spring 2018)

PHYS 530A: Quantum Mechanics II

Due date: Tuesday, 2018 Feb 6, 4.30pm

1. (**20 points.**) (Based on Milton's lecture notes.) The vector product is defined by

$$(\mathbf{A} \times \mathbf{B})_3 = A_1 B_2 - A_2 B_1, \tag{1}$$

and similarly (cyclically) for the 1 and 2 components. If $A_{1,2,3}$ are elements of a non-commutative algebra, the components of $\mathbf{A}$ do not commute, and $\mathbf{A} \times \mathbf{A} \neq 0$ in general. Show that

$$\frac{\boldsymbol{\sigma}}{2} \times \frac{\boldsymbol{\sigma}}{2} = i \frac{\boldsymbol{\sigma}}{2}. \tag{2}$$

This is a special case of the angular momentum statement

$$\mathbf{J} \times \mathbf{J} = i\hbar \mathbf{J}. \tag{3}$$

2. (**20 points.**) (Based on Milton's lecture notes.) Consider a rotation of the coordinate system about the z-axis through an angle ϕ:

$$x' = x \cos\phi + y \sin\phi, \tag{4a}$$
$$y' = -x \sin\phi + y \cos\phi, \tag{4b}$$
$$z' = z. \tag{4c}$$

Pauli matrices, $\boldsymbol{\sigma}$, transform like a vector. The components of Pauli matrices transform as

$$\sigma_{x'} = \sigma_x \cos\phi + \sigma_y \sin\phi, \tag{5a}$$
$$\sigma_{y'} = -\sigma_x \sin\phi + \sigma_y \cos\phi, \tag{5b}$$
$$\sigma_{z'} = \sigma_z. \tag{5c}$$

Verify that the transformed components of Pauli matrices have the same algebraic properties as the original components:

$$\sigma_{x'}^2 = \sigma_{y'}^2 = 1, \qquad \sigma_{x'} \sigma_{y'} = i \sigma_{z'}. \tag{6}$$

3. (**20 points.**) Evaluate

$$[(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b})]\,(\boldsymbol{\sigma} \cdot \mathbf{c}). \tag{7}$$

Then evaluate

$$(\boldsymbol{\sigma} \cdot \mathbf{a})\,[(\boldsymbol{\sigma} \cdot \mathbf{b})(\boldsymbol{\sigma} \cdot \mathbf{c})]\,. \tag{8}$$

Are they equal?