Homework No. 05 (Spring 2018)

PHYS 530A: Quantum Mechanics II

Due date: Tuesday, 2018 Feb 20, 4.30pm

1. (**40 points.**) An eigenbasis that spans an n-dimensional space consists of eigenvectors $\hat{\mathbf{e}}_i$, where $i = 1, 2, \ldots, n$. These eigenvectors have n components that can be indexed using $a, b = 1, 2, \ldots, n$. That is, $\hat{\mathbf{e}}_i = \mathrm{e}_i^a$. Thus, using Einstein summation convention, the orthonormality conditions can be stated as
$$\hat{\mathbf{e}}_i^\dagger \cdot \hat{\mathbf{e}}_j = \delta_{ij}, \qquad \text{or} \qquad \mathrm{e}_i^{a\dagger} \mathrm{e}_j^a = \delta_{ij}, \tag{1}$$
and the completeness relation can be stated as
$$\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1^\dagger + \ldots + \hat{\mathbf{e}}_n \hat{\mathbf{e}}_n^\dagger = \mathbf{1}, \qquad \text{or} \qquad \mathrm{e}_i^a \mathrm{e}_i^{b\dagger} = \delta^{ab}. \tag{2}$$
In this spirit, consider the following eigenbasis, constructed using n-th roots of unity,
$$\mathrm{e}_k^l = \frac{1}{\sqrt{n}} (u_k')^l = \frac{1}{\sqrt{n}} e^{i \frac{2\pi}{n} kl}, \qquad k, l = 1, 2, \ldots, n. \tag{3}$$
Show that
$$\sum_{k=1}^{n} e^{i \frac{2\pi}{n} k(l-l')} = n \, \delta_{ll'}, \tag{4}$$
and using this relation verify that the eigenbasis satisfies the completeness and orthonormality relations. For $n = 2$, the eigenvectors are
$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \hat{\mathbf{e}}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5}$$
Show that these eigenvectors satisfy the orthonormality and completeness relations. Determine the eigenvectors for $n = 3$ and verify the corresponding completeness and orthonormality relations. Caution: Do not forget the complex conjugation.

2. (**20 points.**) Show that the action of a unitary operator U on a function $f(A)$, where A is an operator, satisfies
$$U f(A) U^{-1} = f(U A U^{-1}). \tag{6}$$

3. (**20 points.**) In terms of the eigenvectors of the complementary variables, U and V, introduced in class, evaluate
$$\sum_{k=1}^{n} \langle v_l' | u_k' \rangle \langle u_k' | v_m' \rangle. \tag{7}$$
Thus, derive
$$\sum_{k=1}^{n} e^{i \frac{2\pi}{n} k(m-l)} = n \delta_{lm}. \tag{8}$$