Homework No. 06 (Spring 2018)

PHYS 530A: Quantum Mechanics II

Due date: Tuesday, 2018 Mar 6, 4.30pm

1. (**100 points.**) Consider a normalized state

$$| \,\rangle = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x_1 + ix_2 \\ y_1 + iy_2 \end{pmatrix}, \qquad |u|^2 + |v|^2 = 1. \tag{1}$$

   (a) Show that the expectation value of the Pauli matrices with respect to the above state satisfies

$$\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 = 1. \tag{2}$$

   Thus, conclude that the magnitude of the expectation value of each of the Pauli matrix is less than or equal to unity.

   (b) Define the errors in the measurement of the Pauli matrices to be

$$(\delta\sigma'_x)^2 = \langle \, |\{\sigma_x - \langle\sigma_x\rangle\}^2| \, \rangle, \tag{3a}$$
$$(\delta\sigma'_y)^2 = \langle \, |\{\sigma_y - \langle\sigma_y\rangle\}^2| \, \rangle, \tag{3b}$$
$$(\delta\sigma'_z)^2 = \langle \, |\{\sigma_z - \langle\sigma_z\rangle\}^2| \, \rangle. \tag{3c}$$

   Show that

$$(\delta\sigma'_x)^2 = 1 - \langle\sigma_x\rangle^2, \tag{4a}$$
$$(\delta\sigma'_y)^2 = 1 - \langle\sigma_y\rangle^2, \tag{4b}$$
$$(\delta\sigma'_z)^2 = 1 - \langle\sigma_z\rangle^2. \tag{4c}$$

   Thus, conclude that the errors in the measurement of each of the Pauli matrix is less than or equal to unity. Show that

$$(\delta\sigma'_x)^2 + (\delta\sigma'_y)^2 + (\delta\sigma'_z)^2 = 2. \tag{5}$$

   (c) Using Robertson's generalization of Heisenberg's uncertainty relation

$$(\delta A)(\delta B) \geq \frac{1}{2}|\langle C\rangle|, \tag{6}$$

   deduce the uncertainty relations for the Pauli matrices to be

$$(\delta\sigma'_x)^2(\delta\sigma'_y)^2 \geq \langle\sigma_z\rangle^2, \tag{7a}$$
$$(\delta\sigma'_y)^2(\delta\sigma'_z)^2 \geq \langle\sigma_x\rangle^2, \tag{7b}$$
$$(\delta\sigma'_z)^2(\delta\sigma'_x)^2 \geq \langle\sigma_y\rangle^2. \tag{7c}$$

Combine these uncertainty relations to derive an uncertainty relation involving all the three Pauli matrices,

$$(\delta\sigma'_x)^2(\delta\sigma'_y)^2 + (\delta\sigma'_y)^2(\delta\sigma'_z)^2 + (\delta\sigma'_z)^2(\delta\sigma'_x)^2 \geq 1. \tag{8}$$

(d) Show that minimal uncertainty in each of the above relations is attained, if

$$\langle\sigma_x\rangle\langle\sigma_y\rangle = 0, \tag{9a}$$

$$\langle\sigma_y\rangle\langle\sigma_z\rangle = 0, \tag{9b}$$

$$\langle\sigma_z\rangle\langle\sigma_x\rangle = 0. \tag{9c}$$

Since the expectation value of the Pauli matrices satisfy Eq. (2) all three of the expectation value of Pauli matrices can not be zero for a given state. But, two of them can be zero, which is sufficient for minimizing all the above uncertainty relations. For example, a state satisfying

$$\langle\sigma_x\rangle = \langle\sigma_y\rangle = 0 \tag{10}$$

minimizes all the four uncertainty relations above.

(e) Show that

$$|\text{min}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{11}$$

is such a state. Evaluate $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$, $\langle\sigma_z\rangle$, $\langle\sigma_x^2\rangle$, $\langle\sigma_y^2\rangle$, $\langle\sigma_z^2\rangle$, $\delta\sigma'_x$, $\delta\sigma'_y$, and $\delta\sigma'_z$, when the system is in this state.

(f) Understand the Stern-Gerlach experiment in this light.