

Homework No. 09 (Spring 2018)

PHYS 530A: Quantum Mechanics II

Due date: Tuesday, 2018 Apr 3, 4.30pm

1. (40 points.) The components of angular momentum \mathbf{J} satisfy the commutation relations

$$\frac{1}{i\hbar}[J_i, J_j] = \varepsilon_{ijk}J_k. \quad (1)$$

The general properties of angular momentum can be deduced from these commutation relations. Since \mathbf{J}^2 is a scalar, it commutes with angular momentum \mathbf{J} . Thus, the common eigenvectors of \mathbf{J}^2 and J_z constitute a suitable set of basis vectors for discussing a dynamical system involving only the angular momentum. Let us denote the eigenvalues of these operators by the labeling scheme $\mathbf{J}^2 = j(j+1)\hbar^2$, and $J_z' = m\hbar$. Thus, we write

$$\frac{1}{\hbar^2}\mathbf{J}^2|j, m\rangle = j(j+1)|j, m\rangle, \quad (2a)$$

$$\frac{1}{\hbar}J_z|j, m\rangle = m|j, m\rangle. \quad (2b)$$

Let us also construct (non-Hermitian) operators

$$J_{\pm} = J_x \pm iJ_y. \quad (3)$$

Observe that $J_{\pm}^{\dagger} = J_{\mp}$.

- (a) Show that

$$\frac{1}{\hbar}J_z\{J_+|j, m\rangle\} = (m+1)\{J_+|j, m\rangle\}. \quad (4)$$

Thus deduce that if m is an eigenvalue of J_z , then $(m+1)$ is also an eigenvalue of J_z . Similarly, show that

$$\frac{1}{\hbar}J_z\{J_-|j, m\rangle\} = (m-1)\{J_-|j, m\rangle\}. \quad (5)$$

Thus deduce that if m is an eigenvalue of J_z , then $(m-1)$ is also an eigenvalue of J_z .

- (b) Show that

$$J_+J_- = \mathbf{J}^2 - J_z^2 + \hbar J_z \quad (6)$$

is a Hermitian operator. A Hermitian operator has real eigenvalues, but, since $J_+J_- = J_-^{\dagger}J_+$, infer further that it has non-negative eigenvalues. Thus, deduce that

$$j(j+1) - m(m-1) \geq 0, \quad (7)$$

and then infer

$$-j \leq m \leq j + 1. \quad (8)$$

Similarly, show how that

$$J_- J_+ = \mathbf{J}^2 - J_z^2 - \hbar J_z \quad (9)$$

is a Hermitian operator, and deduce that

$$j(j+1) - m(m+1) \geq 0, \quad (10)$$

and then infer

$$-j - 1 \leq m \leq j. \quad (11)$$

Using Eqs. (8) and (11) in conjunction, show that

$$-j \leq m \leq j. \quad (12)$$

(Refer Sec. 36 Dirac's QM book.)

(c) Using Eq. (12) infer that

$$j \geq 0. \quad (13)$$

Note that \mathbf{J}^2 being the square of a Hermitian operator implies $j(j+1) \geq 0$, but it does not imply that j should be non-negative.

(d) The $2j$ transitions from $m = -j$ to $m = j$ happen in $n = 0, 1, 2, \dots$ steps. Thus, $2j = n$. Thus, conclude that

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \quad (14a)$$

$$m = -j, -j+1, \dots, j. \quad (14b)$$