Homework No. 11 (Spring 2018)

PHYS 530A: Quantum Mechanics II

Due date: Tuesday, 2018 May 8, 5.00pm

1. (**100 points.**) The angular momentum can be decomposed as

$$\mathbf{J} = \mathbf{S} + \mathbf{L}, \tag{1}$$

where $\mathbf{S}$ is the spin or internal angular momentum, and $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the orbital or external angular momentum. For the case $\mathbf{S} = 0$ the eigenvalues of angular momentum are necessarily integer valued, because $\mathbf{r} \cdot \mathbf{L} = 0$. Let us denote the eigenvalues by the labeling scheme $\mathbf{L}'^2 = \hbar^2 l(l+1)$ and $L'_z = \hbar m$, such that

$$\mathbf{L}^2 |l, m\rangle = \hbar^2 l(l+1)|l, m\rangle, \tag{2a}$$

$$L_z |l, m\rangle = \hbar m |l, m\rangle, \tag{2b}$$

where

$$l = 0, 1, 2, \ldots, \tag{3a}$$

$$m = -l, -l+1, \ldots, l. \tag{3b}$$

The eigenvectors of orbital angular momentum are suitably realized by functions on the surface of a unit sphere, coordinated by spherical polar coordinates θ' and ϕ' or the unit vector $\hat{\mathbf{r}}'$. These wavefunctions defined using the projections

$$\langle \hat{\mathbf{r}}' | l, m\rangle = Y_{lm}(\theta', \phi') \tag{4}$$

are the spherical harmonics.

(a) Show that in the position basis, here restricted to the surface of a unit sphere, we have

$$\langle \hat{\mathbf{r}}' | \mathbf{L} | \ \rangle = \langle \hat{\mathbf{r}}' | \mathbf{r} \times \mathbf{p} | \ \rangle = \frac{\hbar}{i} (\mathbf{r}' \times \boldsymbol{\nabla}') \langle \hat{\mathbf{r}}' | \ \rangle. \tag{5}$$

Using Eq. (5) in Eqs. (2) show that the differential equations for spherical harmonics are given by

$$-(\mathbf{r}' \times \boldsymbol{\nabla}') \cdot (\mathbf{r}' \times \boldsymbol{\nabla}') Y_{lm}(\theta', \phi') = l(l+1) Y_{lm}(\theta', \phi'), \tag{6a}$$

$$\frac{1}{i} \hat{\mathbf{z}}' \cdot (\mathbf{r}' \times \boldsymbol{\nabla}') Y_{lm}(\theta', \phi') = m Y_{lm}(\theta', \phi'). \tag{6b}$$

(b) Show that the raising and lowering operators defined using

$$L_\pm = L_x \pm iL_y, \tag{7}$$

leading to raising and lowering operations

$$L_\pm|l,m\rangle = \hbar\sqrt{(l\mp m)(l\pm m+1)}|l,m\pm 1\rangle, \tag{8}$$

correspond to the differential equations

$$\frac{1}{i}\Big[\hat{\mathbf{x}}'\cdot(\mathbf{r}'\times\boldsymbol{\nabla}')\pm i\hat{\mathbf{y}}'\cdot(\mathbf{r}'\times\boldsymbol{\nabla}')\Big]Y_{lm}(\theta',\phi') = \sqrt{(l\mp m)(l\pm m+1)}Y_{l,m\pm 1}(\theta',\phi'). \tag{9}$$

(c) Using the differential operator in spherical polar coordinates,

$$\boldsymbol{\nabla}' = \hat{\mathbf{r}}'\frac{\partial}{\partial r'} + \hat{\boldsymbol{\theta}}'\frac{1}{r'}\frac{\partial}{\partial\theta'} + \hat{\boldsymbol{\phi}}'\frac{1}{r'\sin\theta'}\frac{\partial}{\partial\phi'}, \tag{10}$$

where

$$\hat{\mathbf{r}}' = \hat{\mathbf{x}}'\sin\theta'\cos\phi' + \hat{\mathbf{y}}'\sin\theta'\sin\phi' + \hat{\mathbf{z}}'\cos\theta', \tag{11a}$$

$$\hat{\boldsymbol{\theta}}' = \hat{\mathbf{x}}'\cos\theta'\cos\phi' + \hat{\mathbf{y}}'\cos\theta'\sin\phi' - \hat{\mathbf{z}}'\sin\theta', \tag{11b}$$

$$\hat{\boldsymbol{\phi}}' = -\hat{\mathbf{x}}'\sin\phi' + \hat{\mathbf{y}}'\cos\phi', \tag{11c}$$

show that

$$\mathbf{r}'\times\boldsymbol{\nabla}' = \hat{\boldsymbol{\phi}}'\frac{\partial}{\partial\theta'} - \hat{\boldsymbol{\theta}}'\frac{1}{\sin\theta'}\frac{\partial}{\partial\phi'} \tag{12a}$$

$$= \hat{\mathbf{x}}'\left[-\sin\phi'\frac{\partial}{\partial\theta'} - \cos\phi'\cot\theta'\frac{\partial}{\partial\phi'}\right] + \hat{\mathbf{y}}'\left[\cos\phi'\frac{\partial}{\partial\theta'} - \sin\phi'\cot\theta'\frac{\partial}{\partial\phi'}\right] + \hat{\mathbf{z}}'\frac{\partial}{\partial\phi'}. \tag{12b}$$

Thus, show the correspondence

$$L_z: \quad \hat{\mathbf{z}}'\cdot\frac{\hbar}{i}(\mathbf{r}'\times\boldsymbol{\nabla}') = \frac{\hbar}{i}\frac{\partial}{\partial\phi'}, \tag{13a}$$

$$L_x: \quad \hat{\mathbf{x}}'\cdot\frac{\hbar}{i}(\mathbf{r}'\times\boldsymbol{\nabla}') = \frac{\hbar}{i}\left[-\sin\phi'\frac{\partial}{\partial\theta'} - \cos\phi'\cot\theta'\frac{\partial}{\partial\phi'}\right], \tag{13b}$$

$$L_y: \quad \hat{\mathbf{y}}'\cdot\frac{\hbar}{i}(\mathbf{r}'\times\boldsymbol{\nabla}') = \frac{\hbar}{i}\left[\cos\phi'\frac{\partial}{\partial\theta'} - \sin\phi'\cot\theta'\frac{\partial}{\partial\phi'}\right]. \tag{13c}$$

Further, verify the correspondence

$$L^2: \quad \frac{\hbar}{i}(\mathbf{r}'\times\boldsymbol{\nabla}')\cdot\frac{\hbar}{i}(\mathbf{r}'\times\boldsymbol{\nabla}') = \frac{\hbar^2}{i^2}\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \frac{1}{\sin^2\theta'}\frac{\partial^2}{\partial\phi'^2}\right], \tag{14a}$$

$$L_z^2: \quad \frac{\hbar^2}{i^2}\Big[\hat{\mathbf{z}}'\cdot(\mathbf{r}'\times\boldsymbol{\nabla}')\Big]^2 = \frac{\hbar^2}{i^2}\frac{\partial^2}{\partial\phi'^2}, \tag{14b}$$

$$L_\pm: \quad \frac{\hbar}{i}\Big[\hat{\mathbf{x}}'\cdot(\mathbf{r}'\times\boldsymbol{\nabla}')\pm i\hat{\mathbf{y}}'\cdot(\mathbf{r}'\times\boldsymbol{\nabla}')\Big] = \frac{\hbar}{i}e^{\pm i\phi}\left[\pm i\frac{\partial}{\partial\theta'} - \cot\theta'\frac{\partial}{\partial\phi'}\right], \tag{14c}$$

(d) Thus, show that the eigenfunctions of angular momentum in the position basis, the spherical harmonics, satisfy the differential equations given by

$$L_z : \qquad \frac{1}{i}\frac{\partial}{\partial\phi'} Y_{lm}(\theta',\phi') = m Y_{lm}(\theta',\phi'), \tag{15a}$$

$$L^2 : \quad -\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \frac{1}{\sin^2\theta'}\frac{\partial^2}{\partial\phi'^2}\right] Y_{lm}(\theta',\phi') = l(l+1) Y_{lm}(\theta',\phi'), \tag{15b}$$

$$L_\pm : \qquad \frac{i}{i} e^{\pm i\phi}\left[\pm i\frac{\partial}{\partial\theta'} - \cot\theta'\frac{\partial}{\partial\phi'}\right] Y_{lm}(\theta',\phi') = \sqrt{(l\mp m)(l\pm m+1)}\, Y_{l,m\pm 1}(\theta',\phi'). \tag{15c}$$

Further, verify

$$L_+L_- : \quad -\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \frac{1}{\sin^2\theta'}\frac{\partial^2}{\partial\phi'^2} - \frac{\partial^2}{\partial\phi'^2} - \frac{1}{i}\frac{\partial}{\partial\phi'}\right] Y_{lm}(\theta',\phi') = \big[l(l+1) - m(m-1)\big] Y_{lm}(\theta',\phi'), \tag{16a}$$

$$L_-L_+ : \quad -\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \frac{1}{\sin^2\theta'}\frac{\partial^2}{\partial\phi'^2} - \frac{\partial^2}{\partial\phi'^2} + \frac{1}{i}\frac{\partial}{\partial\phi'}\right] Y_{lm}(\theta',\phi') = \big[l(l+1) - m(m+1)\big] Y_{lm}(\theta',\phi'), \tag{16b}$$

$$L_x^2 + L_y^2 : \quad -\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \cot^2\theta'\frac{\partial^2}{\partial\phi'^2}\right] Y_{lm}(\theta',\phi') = \big[l(l+1) - m^2\big] Y_{lm}(\theta',\phi'). \tag{16c}$$