Homework No. 13 (Spring 2018)

PHYS 530A: Quantum Mechanics II

Due date: Thursday, 2018 May 3, 4.30pm

1. (**20 points.**) The components of the position and momentum operator, $\mathbf{r}$ and $\mathbf{p}$, respectively, satisfy the commutation relations $[r_i, p_j] = i\hbar\delta_{ij}$. Verify the following:
   (a) $\mathbf{r} \times \mathbf{p} + \mathbf{p} \times \mathbf{r} = 0$.
   (b) $\mathbf{r} \cdot \mathbf{p} - \mathbf{p} \cdot \mathbf{r} = 3i\hbar$.
   (c) $(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{p}) - (\mathbf{b} \cdot \mathbf{p})(\mathbf{a} \cdot \mathbf{r}) = i\hbar(\mathbf{a} \cdot \mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ and numerical.
   (d) $\mathbf{r} \times (\mathbf{r} \times \mathbf{p}) = \mathbf{r}\,\mathbf{p} \cdot \mathbf{r} - \mathbf{p}r^2 + i\hbar\mathbf{r}$.

2. (**20 points.**) Using commutation relations between $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{L}$, verify the following:
   (a) $\mathbf{L} \times \mathbf{L} = i\hbar\mathbf{L}$.
   (b) $\mathbf{p} \times \mathbf{L} + \mathbf{L} \times \mathbf{p} = 2i\hbar\mathbf{p}$.
   (c) $-\mathbf{L} \times \mathbf{p} \cdot \frac{\mathbf{r}}{r} = L^2 \frac{1}{r} = \frac{1}{r} L^2$.

3. (**30 points.**) Using commutation relations between $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{L}$, verify the relation

$$\mathbf{p} \times \mathbf{L} \cdot \mathbf{p} = 2i\hbar\, p^2. \tag{1}$$

Thus, verify that either of the three equalities for

$$\mathbf{M} = -\frac{1}{2}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) = -\mathbf{p} \times \mathbf{L} + i\hbar\mathbf{p} = \mathbf{L} \times \mathbf{p} - i\hbar\mathbf{p} \tag{2}$$

leads to

$$M^2 = (L^2 + \hbar^2)p^2. \tag{3}$$

This ensures that either of the following three expressions for the Axial vector

$$\begin{aligned} \mathbf{A} &= \hat{\mathbf{r}} - \frac{1}{\mu Z e^2}\frac{1}{2}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) && \text{(4a)} \\ &= \hat{\mathbf{r}} - \frac{1}{\mu Z e^2}\mathbf{p} \times \mathbf{L} + \frac{i\hbar}{\mu Z e^2}\mathbf{p} && \text{(4b)} \\ &= \hat{\mathbf{r}} + \frac{1}{\mu Z e^2}\mathbf{L} \times \mathbf{p} - \frac{i\hbar}{\mu Z e^2}\mathbf{p} && \text{(4c)} \end{aligned}$$

leads to

$$A^2 = 1 + \frac{2(L^2 + \hbar^2)H}{\mu Z^2 e^4}. \tag{5}$$

After using the Bohr quantization condition $L = n' \hbar$, where $n' = 0, 1, 2, \ldots$, and presuming that the orbit is a circle that has eccentricity $A = 0$, Eq. (5) leads to the energy levels predicted by the Bohr model,

$$H = -\frac{\mu Z^2 e^4}{\hbar^2} \frac{1}{2n^2}, \qquad n = 1, 2, 3, \ldots . \tag{6}$$

Show that the (classical) analysis of hydrogen atom, (that does not accommodate the Heisenberg uncertainty relation contained in the commutation relations between **r** and **p**,) in addition, admits the $n = 0$ energy state which permits orbits of vanishing radius.