Midterm Exam No. 02 (Spring 2019)

PHYS 520B: Electromagnetic Theory

Date: 2019 Apr 2

1. (**20 points.**) Evaluate the integral

$$\lim_{\epsilon \to 0+} \int_{0-\epsilon}^{\infty} dx \, \frac{s^{\frac{x}{\pi}} \, \delta(\sin x)}{\Gamma\left(\frac{x}{\pi} + 1\right)} \tag{1}$$

as a sum. Recognize the sum as an elementary function.

2. (**20 points.**) The electric and magnetic field generated by a particle with charge q moving along the z axis with speed v, $\beta = v/c$, can be expressed in the form

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{\left[x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z - vt)\hat{\mathbf{k}}\right]}{(x^2 + y^2)} \frac{(x^2 + y^2)(1 - \beta^2)}{\left[(x^2 + y^2)(1 - \beta^2) + (z - vt)^2\right]^{\frac{3}{2}}}, \tag{2a}$$

$$c\mathbf{B}(\mathbf{r}, t) = \boldsymbol{\beta} \times \mathbf{E}(\mathbf{r}, t). \tag{2b}$$

(a) Consider the distribution

$$\delta(s) = \lim_{\epsilon \to 0} \frac{1}{2} \frac{\epsilon}{(\epsilon + s^2)^{\frac{3}{2}}}. \tag{3}$$

Show that

$$\delta(s) \begin{cases} \to \frac{1}{\sqrt{\epsilon}} \to \infty, & \text{if} \quad s = 0, \\ \to \frac{\epsilon}{s^2} \to 0, & \text{if} \quad s \neq 0. \end{cases} \tag{4}$$

Further, show that

$$\int_{-\infty}^{\infty} ds \, \delta(s) = 1. \tag{5}$$

(b) Thus, verify that the electric and magnetic field of a charge approaching the speed of light can be expressed in the form

$$\mathbf{E}(\mathbf{r}, t) = \frac{2q}{4\pi\varepsilon_0} \frac{\hat{\boldsymbol{\rho}}}{\rho} \delta(z - ct), \tag{6a}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \frac{2q}{4\pi\varepsilon_0} \frac{\hat{\boldsymbol{\phi}}}{\rho} \delta(z - ct) = 2cq \frac{\mu_0}{4\pi} \frac{\hat{\boldsymbol{\phi}}}{\rho} \delta(z - ct), \tag{6b}$$

where $\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and $\rho = \sqrt{x^2 + y^2}$. These fields are confined on the $z = ct$ plane moving with speed c.

(c) To confirm that the above confined fields are indeed solutions to the Maxwell equations, verify the following:

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} q\delta^{(2)}(\boldsymbol{\rho})\delta(z - ct), \tag{7a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{7b}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{7c}$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 qc\hat{\mathbf{z}}\delta^{(2)}(\boldsymbol{\rho})\delta(z - ct). \tag{7d}$$

This is facilitated by writing

$$\nabla = \nabla_\rho + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \tag{8}$$

and accomplished by using the following identities:

$$\nabla_\rho \cdot \left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 2\pi\delta^{(2)}(\boldsymbol{\rho}), \qquad \nabla_\rho \times \left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 0, \tag{9a}$$

$$\nabla_\rho \cdot \left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = 0, \qquad \nabla_\rho \times \left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = 2\pi\delta^{(2)}(\boldsymbol{\rho}). \tag{9b}$$