Homework No. 02 (Spring 2019)

PHYS 520B: Electromagnetic Theory

Due date: Thursday, 2019 Feb 7, 4.30pm

1. (**20 points.**) A homogeneous magnetic field $\mathbf{B}$ is characterized by the vector potential

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}. \tag{1}$$

   (a) Evaluate $\boldsymbol{\nabla} \times \mathbf{A}$.

   (b) Verify that this construction satisfies the radiation gauge by showing that

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{2}$$

   (c) Is this construction unique? No. Remember the freedom of gauge transformation,

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\lambda(\mathbf{r}, t), \tag{3}$$

   where $\lambda(\mathbf{r}, t)$ is an arbitrary function. Show that for any given vector potential $\mathbf{A}$ there exists λ that satisfies

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\nabla^2 \lambda \tag{4}$$

   that leads to the construction of $\mathbf{A}'$ satisfying the radiation gauge.

   (d) Let us consider the special case when $\mathbf{B} = B\,\hat{\mathbf{z}}$.

      i. Show that, for this case,

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} = -\frac{1}{2}By\,\hat{\mathbf{i}} + \frac{1}{2}Bx\,\hat{\mathbf{j}} = \frac{1}{2}Br\,\hat{\boldsymbol{\phi}}. \tag{5}$$

      Visualize $\mathbf{A}$ diagramatically.

      ii. Show that

$$\mathbf{A} = 0\,\hat{\mathbf{i}} + Bx\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{z}} \tag{6}$$

      is a satisfactory vector potential for homogeneous magnetic field. Visualize $\mathbf{A}$ diagramatically. But, show that this construction does not satisfy the radiation gauge. Using

$$\lambda(\mathbf{r}, t) = \frac{1}{2}Bxy \tag{7}$$

      construct $\mathbf{A}'$ that satisfies the radiation gauge.

iii. Show that

$$\mathbf{A} = -By\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{z}} \tag{8}$$

is also a satisfactory vector potential for homogeneous magnetic field. Visualize $\mathbf{A}$ diagramatically. Choose a suitable $\lambda(\mathbf{r}, t)$ to construct $\mathbf{A}'$ that satisfies the radiation gauge.

iv. Chose an arbitray $\lambda(\mathbf{r}, t)$, of your choice, to construct another satisfactory vector potential for homogeneous magnetic field.

2. (**20 points.**) The vector potential for a point magnetic moment $\mathbf{m}$ is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3}. \tag{9}$$

Verify that the magnetic field due to the point dipole given by

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{10}$$

satisfies

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{11}$$

3. (**20 points.**) A solenoid has the geometry of a right circular cylinder of radius a and height extending to infinity on both ends. Using Ampere's law show that the magnetic field is uniform inside the solenoid and zero outside the solenoid. How does this result change for a solenoid of arbitrary cross section. Refer literature and reproduce the conclusion.