

Homework No. 02 (2019 Fall)

PHYS 301: Theoretical Methods in Physics

Due date: None

1. Keywords: Hyperbolic functions, trigonometric functions in the complex plane, inverse hyperbolic functions.
2. Hyperbolic cosine function and sine function are defined using the exponential function. We have

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad (1a)$$

$$\sinh x = \frac{e^x - e^{-x}}{2}. \quad (1b)$$

Here, and in the following, assume x and y to be real. Recall that the corresponding trigonometric functions are defined in terms of the exponential function as

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad (2a)$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}. \quad (2b)$$

Hyperbolic functions extend the domain of the corresponding trigonometric functions to the complex plane.

- (a) Show that

$$\cosh x = \cos(ix), \quad (3a)$$

$$\sinh x = -i \sin(ix). \quad (3b)$$

- (b) Plot $\cosh x$ and $\sinh x$ as functions of x .

- (c) Using Eqs.(1) derive the identity

$$\cosh^2 x - \sinh^2 x = 1. \quad (4)$$

Derive the identities for the sum of arguments of hyperbolic functions,

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y, \quad (5a)$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y. \quad (5b)$$

Derive the derivative operations on hyperbolic functions,

$$\frac{d}{dx} \cosh x = \sinh x, \quad (6a)$$

$$\frac{d}{dx} \sinh x = \cosh x, \quad (6b)$$

and the integral operations,

$$\int dx \cosh x = \sinh x, \quad (7a)$$

$$\int dx \sinh x = \cosh x. \quad (7b)$$

(d) To find the inverse hyperbolic function of sine let us define

$$y = \sinh^{-1} x. \quad (8)$$

Then, we have

$$x = \sinh y = \frac{e^y - e^{-y}}{2}, \quad (9)$$

which can be rewritten as a quadratic equation in e^y ,

$$(e^y)^2 - 2x(e^y) - 1 = 0, \quad (10)$$

with solutions

$$e^y = x \pm \sqrt{x^2 + 1}. \quad (11)$$

Presuming y to be real argue that only one of the roots is consistent with $e^y > 0$. Taking logarithm we have

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}). \quad (12)$$

Similarly, derive the expression

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}). \quad (13)$$